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EXERCISES

IN

ANALYTICAL GEOMETRY.

EXERCISES

IN

ANALYTICAL GEOMETRY

BY

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WITH ILLUSTRATIONS.

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PREFACE.

THESE Exercises have been written with the object of imparting a sound knowledge of the principles of Modern Analytical Geometry to any boys in my more advanced class, who are preparing for Scholarships at the Universities or for the Indian Civil Examination, and it is hoped that the book will be useful to boys doing similar work in other schools.

Candidates for Mathematical Honours at the Universities, who have not devoted much attention to this important subject, would also be benefited by a course of the more difficult problems.

I have to thank my friend Mr C. Leudesdorf, M.A., Fellow of Pembroke College, Oxford, for reading through the proof-sheets and for verifying results.

My acknowledgements are also due to him and to other friends who have kindly furnished me with problems for insertion.

J. M. DYER.

CHELTENHAM COLLEGE,
March 10, 1881.

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EXERCISES

IN

ANALYTICAL GEOMETRY.

I. CO-ORDINATES, &c.

[*The axes are supposed rectangular unless the contrary is stated.*]

1. Find the distance between the points (a, b), and $(a+33, b+56)$.

2. Find the distance from $(-3, 7)$ to $(6, -5)$ when the axes are at right angles; and also when they are inclined at an angle of 60°.

3. The points $(a+r\cos\theta,\ b+r\sin\theta)$, $(a+r\cos\phi,\ b+r\sin\phi)$ are equidistant from the point (a, b).

4. The points $(a, 2a)$, $(a, 3a)$, $\left(a+\frac{\sqrt{3}}{2}a,\ \frac{5a}{2}\right)$ form an equilateral triangle, whose side is a.

5. Shew that the following sets of points will form right-angled triangles, and find the lengths of their hypothenuses:

(i) $(23, 31), (63, 31), (23, 40)$.

(ii) $(a, b), (a+28\sqrt{2}, b+28\sqrt{2}), \left(a+\frac{33}{\sqrt{2}}, b-\frac{33}{\sqrt{2}}\right)$.

6. The points $(0, 0)$; $(a\cos 15^\circ, a\sin 15^\circ)$; and $\{b\cos(75^\circ+\alpha), b\sin(75^\circ+\alpha)\}$ will form an isosceles triangle of perimeter $a+2b$, if

$$\cos(60+\alpha)=\frac{a}{2b}.$$

7. Shew that the points $(3, 0), (0, 3\sqrt{3}), (6, 3\sqrt{3})$ form an equilateral triangle, and find its area.

8. Determine in square inches the area of the triangle whose angles are at the points $(7, 2), (4, -5), (-3, 1)$, measured on the half-inch scale.

9. The perimeter of the quadrilateral whose vertices, taken in order, are

$$\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right), \left(\frac{-b\sqrt{3}}{2}, \frac{b}{2}\right), \left(\frac{-c}{2}, \frac{-c\sqrt{3}}{2}\right),$$

$$\left(\frac{d\sqrt{3}}{2}, -\frac{d}{2}\right),$$

is $\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+d^2}+\sqrt{d^2+a^2}$.

10. Find the co-ordinates of the point of intersection of the lines joining each vertex of a triangle to the middle point of the opposite side; the co-ordinates of whose angular points are given.

11. Find the co-ordinates of the middle point of the line joining the middle points of a pair of opposite sides of any quadrilateral; the co-ordinates of whose angular points are given.

Hence shew that the lines joining the middle points of pairs of opposite sides of a quadrilateral bisect one another.

12. The point in (11) coincides with the middle point of the line joining the middle points of the diagonals of the quadrilateral.

13. OBC is a right-angled triangle, O being the origin, B the right angle, $OB = 56$, $BC = 33$, and OC makes an angle of 30° with the axis of x; find the co-ordinates of B and C.

14. Shew that the distance between the points whose polar co-ordinates are

$$\left(\frac{2a}{1+\cos\theta}, \theta\right), \left(\frac{2a}{1+\cos\phi}, \phi\right),$$

is equal to

$$a\left\{\sec^2\frac{\theta}{2} + \sec^2\frac{\phi}{2} - \cos(\theta - \phi)\right\},$$

provided that

$$\cos^2(\theta - \phi) + 2\cos(\theta - \phi)\left\{\sec^2\frac{\theta}{2}\sec^2\frac{\phi}{2} - \sec^2\frac{\theta}{2} - \sec^2\frac{\phi}{2}\right\} + 2\sec^2\frac{\theta}{2}\sec^2\frac{\phi}{2} = 0.$$

15. Find the radius of the circle circumscribing the triangle whose polar co-ordinates are $(0, 0)$, (r_1, θ_1), (r_2, θ_2).

16. AB is a straight line, O the pole; the bisector of the angle AOB meets AB in C. Shew that

$$OC = \frac{2 . OA . OB}{OA + OB} . \cos COB.$$

II. TRANSFORMATION OF CO-ORDINATES.

1. Transform $xy = 0$ and $x^2 - y^2 = 0$ to a new origin at $(2, 3)$ and to axes making an angle of 30° with the old.

2. $x - y = 4$ becomes $y + 2\sqrt{3} = 0$ when referred to rectangular axes. Find the angle between the former axes, and the angle between the old and new axes of x.

3. Transform $y^2 - 4ax = 0$ to the point $(am^2, 2am)$ as origin and to new axes making an angle $\cot^{-1} m$ with the old.

4. Transform $\frac{x^2}{m^2} + \frac{y^2}{l^2} - 1 = 0$ to a point $(0, l)$ as origin and from axes inclined at an angle ω to rectangular axes, the axis of x remaining the same in direction.

III. THE STRAIGHT LINE.

1. Find the equation of the straight line which joins the intersection of the lines $y = 7x + 3$ and $3y - 5x = 2$, to that of the straight lines $5y + 4x = 12$ and $6y + 8x = 4$.

2. Find the equation of the straight line which passes through the point $(3, -2)$ and is perpendicular to the line which passes through the points $(-5, 7)$ and $(2, 5)$.

3. Find the perpendicular distance of the intersection of the lines $2y - 4x = 10$ and $3y + 9x = -21$ from the line $y - 3x = 15$.

4. Find the equation of the straight line which passes through the point $(7, -5)$, and makes an angle of 45° with the line whose equation is $2y - 6x = 3$.

5. Find the length of the perpendicular drawn from the point $(8, 4)$ to the line $y = 2x - 16$.

6. Find the angle included between the lines whose equations are $2x + 5 = y$, and $4y + 3x = 0$.

7. Shew that the straight line which passes through (h_1, k_1), (h_2, k_2) intersects the straight line through $(k_1, -h_1)$, $(k_2, -h_2)$ at right angles; and find their point of intersection, when $h_1 = 5$, $k_1 = 3$, $h_2 = -4$, $k_2 = 2$.

8. Find the equation to the line which passes through the intersection of the lines $x - 2y = a$, $x + 3y = 2a$, and is parallel to $3x + 4y = 0$. (See Note to III. 1. Answers).

9. In

$$y = 5x - 7 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$y = 7x - 5 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$y = -4x + 2 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

find the length of the portion of (3) intercepted between (1) and (2).

10. Find the points at which the line

$$7y - 21x + 28 = 0$$

cuts the co-ordinate axes, and the angle which it makes with the line which passes through the points $(3, -2)$ and $(-4, 7)$.

11. Find the angle between the lines

$$(m'^2 - mm')\,y = (mm' + m^2)\,x + c,$$
$$(mm' + m'^2)\,y = (mm' - m^2)\,x + c'.$$

12. The straight lines

$$\begin{aligned}3x + 5y &= 27,\\ 4x - 11y &= 31,\\ 18x - 23y &= 147,\end{aligned}$$

pass through the same point.

13. The straight lines

$$\begin{aligned}(a-c)\,x + (b-d)\,y - e &= 0,\\ (a+c)\,x + (b+d)\,y - e &= 0,\\ ax + by - e &= 0,\\ cx + dy &= 0,\end{aligned}$$

pass through the point

$$\frac{x}{-d} = \frac{y}{c} = \frac{e}{-ad+bc}.$$

14. If ω be the angle between the axes, find the tangent of the angle between the lines $y = mx$, $y = m'x$.

15. By the last question shew that $y + x = 0$ and $y - x = 0$ represent two lines at right angles, whatever be the inclination of the axes.

16. On two straight lines intersecting at O points P, P'; Q, Q' are taken two on each line. If OP be a; OP', a'; OQ, b; OQ', b', shew that the equation of the line joining the intersections of PQ, $P'Q'$; $P'Q$, PQ', is

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2.$$

See III. 1, Note.

17. Find in (16) the equations of the lines joining the intersections of PQ, $P'Q'$, and of $P'Q$, PQ' to the origin.

18. O is a fixed point; OB and OC are two straight lines whose direction is given; B and C points such that $OB + OC$ is constant and equal to l. Find the locus of M, a point dividing BC in the ratio $m : n$.

19. If a straight line makes intercepts a and b on rectangular axes, and if α be its inclination to the axis of x, shew that any point (x, y) on the straight line satisfies the condition

$$\frac{1}{(x \sin \alpha - y \cos \alpha)^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

20. The angle between the axes being ω, from a given point $P(a, b)$ perpendiculars are drawn to the axes, PA to the axis of x meeting the axis of y in B'; PB to the axis of y meeting the axis of x in A'; find the lengths AA', BB'.

21. In $x \cos \alpha + y \sin \alpha = p$, if $p = -\frac{\sin^2 \alpha}{\cos \alpha}$, shew that the perpendiculars on this straight line from the points $(m^2, 2m)$, $(mm', m + m')$, $(m'^2, 2m')$ form a geometrical progression.

22. If p_1, p_2, p_3 denote the perpendiculars from the point (h, k) on the straight lines

$$ax + by - hk = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(a - b) x + (a + b) y - hk = 0 \;\ldots\ldots\ldots\ldots (2),$$

$$(a + b) x - (a - b) y - hk = 0 \;\ldots\ldots\ldots\ldots (3),$$

respectively, shew that

(i) $p_2 + p_3 = \sqrt{2} p_1$,

(ii) $p_1^2 - 2p_2 p_3 = \frac{(bh - ak)^2}{a^2 + b^2}$.

23. The following equations represent straight lines; find them, and the angles they include:

(1) $6y^2 - xy - x^2 + 30y + 36 = 0,$

(2) $y^2 - 4xy + x^2 + 9y - (18 + \sqrt{3})\,x + 20 = 0,$

(3) $abx^2 + (b^2 - a^2)\,xy - aby^2 - c\,(a + b)\,x$
$- c\,(b - a)\,y + c^2 = 0.$

24. From a fixed point straight lines are drawn cutting the axes in P, Q; P', Q'; find the locus of the intersection of PQ' and $P'Q$.

25. $(a^2 + b^2 - 1)(x^2 + y^2 - 1) = (ax + by - 1)^2$ represents two straight lines. Find them and their point of intersection.

26. If (x_1, y_1), (x_2, y_2), (x_3, y_3) be the co-ordinates of the vertices of a triangle ABC, shew that

$$\tan A = \frac{(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)}{(x_3 - x_1)(x_2 - x_1) + (y_2 - y_1)(y_3 - y_1)}.$$

27. Find the equations of the lines bisecting the sides of the triangle in (26) at right angles.

28. Find the co-ordinates of the centre of the circumscribing circle of the triangle in (26).

29. Given three lines, if $\tan^{-1} m_1$ be the angle between the first and second, $\tan^{-1} m_2$, $\tan^{-1} m_3$ the angles between the second and third, and the third and first respectively, measured in the same direction, then

$$m_1 + m_2 + m_3 = m_1 m_2 m_3.$$

30. Two limited straight lines OA and OB are at right angles to one another. From P, the point where OA

and OB subtend equal angles, perpendiculars p and q are drawn to OA, OB respectively; shew that

$$\frac{OA^2 . p^2}{OB^2 . q^2} = \frac{p^2 + (OA - q)^2}{q^2 + (OB - p)^2}.$$

31. OB, OC are fixed lengths, a, and b, cut off on axes inclined at any angle. In BC any point $H (h, k)$ is taken. Any straight line through H cuts the axes in A and D respectively. Find the locus of intersection of the lines joining the middle points of pairs of opposite sides of the quadrilateral $ABCD$.

32. Find the area contained by the lines

(1) $$x \cos \alpha + y \sin \alpha = a,$$
$$x \cos \alpha - y \sin \alpha = a,$$
$$(x + y) \cos \alpha - (x - y) \sin \alpha = a.$$

(2) $$x \cos \alpha + y \sin \alpha = a,$$
$$x \sin \alpha - y \cos \alpha = a,$$
$$(x + y) \cos \alpha - (x - y) \sin \alpha = a.$$

33. Find the area contained by the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a} = \frac{y}{2b}, \quad \frac{x}{a} = \frac{y}{3b}.$$

34. If Δ, Δ' be the areas of triangles whose angular points are $\{(b, c); (c, a); (a, b)\}$ and $\{(ac - b^2, ab - c^2); (ba - c^2, bc - a^2); (cb - a^2, ca - b^2)\}$ respectively; shew that

$$\Delta' = \Delta (a + b + c)^2.$$

35. The area of the triangle whose vertices are (0, 0), $(3\sin^3\theta, -1)$, $(4, 27\sin\theta)$ is equal to twice the product of the areas of the triangles whose vertices are

$$\{(0, 0),\ (9\sin^2\theta, 2),\ (3\sin\theta - 1, 1)\},$$
$$\{(0, 0),\ (9\sin^2\theta, -2),\ (3\sin\theta + 1, 1)\}.$$

36. If the vertices of a triangle whose area is Δ_1 be

$$\{(0, 0),\ (a\tan\theta,\ b\cot\theta),\ (a\sin\theta, b\cos\theta)\},$$

of Δ_2

$$\{(a, b), (a\sec^2\theta, b\operatorname{cosec}^2\theta), (a + a\sin^2\theta, b + b\cos^2\theta)\}$$

of Δ_3

$$\{(0, 0),\ (a\tan\theta,\ -b\cot\theta),\ (a\sin\theta, b\cos\theta)\},$$

then
$$\Delta_1\Delta_3 = \frac{ab}{2}\Delta_2.$$

37. PQ cuts off intercepts a and b from axes intersecting at O, and inclined at any angle. $P'Q'$ cuts off intercepts al, $b(1 - l)$, from the axes, l being less than unity. Find the co-ordinates of a point A on $P'Q'$ such that the triangle $OP'Q'$ may be one half the triangle APQ.

38. Find the bisectors of the angles contained by

$$y - b = \frac{2m}{1 - m^2}(x - a) \text{ and } y - b = \frac{2m'}{1 - m'^2}(x - a).$$

39. Shew that

$$bx\cos\theta + ay\sin\theta = ab,$$
$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\cos\theta\sin\theta,$$

are the bisectors of the angles between the two lines

$$x - a\cos\theta = \frac{a\cos\theta \pm \sqrt{a^2 - b^2}}{b\sin\theta}(y - b\sin\theta).$$

IV. PAIRS OF STRAIGHT LINES.

To find the angle between the straight lines

$$ax^2 + 2hxy + by^2 = 0.$$

Divide through by b and put

$$y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = (y - mx)(y - m'x),$$

whence $$m + m' = -\frac{2h}{b}, \quad mm' = \frac{a}{b},$$

and $$m' - m = \frac{2\sqrt{h^2 - ab}}{b}.$$

But the tangent of the angle between the lines $y - mx = 0$, $y - m'x = 0$ is

$$\frac{(m' - m)\sin\omega}{1 + mm' - (m + m')\cos\omega}$$

if the angle between the axes is ω (see III. 14); which becomes

$$\frac{2\sqrt{h^2 - ab}\sin\omega}{a + b - 2h\cos\omega}.$$

For rectangular axes this reduces to

$$\frac{2\sqrt{h^2 - ab}}{a + b}.$$

The condition of perpendicularity in the two cases is $a + b = 2h\cos\omega$, and $a = -b$ respectively.

The above method of resolving the equations of the lines into factors, and then substituting for $m + m'$, mm' and $m' - m$ their values, will be found useful in the following examples.

1. The area cut off by $y = mx + c$, from

$$ax^2 + 2hxy + by^2 = 0,$$

is

$$\frac{c^2 \sqrt{h^2 - ab}}{a + 2hm + bm^2}.$$

2. The product of the perpendiculars from (x', y') on the pair of lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a-b)^2 + 4h^2}}.$$

3. $lx + my + n = 0$ cuts the straight lines

$$ax^2 + 2hxy + by^2 = 0,$$

making angles α and β with them. Shew that

$$\tan\alpha + \tan\beta = 2\,\frac{(a-b)\,lm - h\,(l^2 - m^2)}{al^2 + 2hlm + bm^2}.$$

4. The area cut by the straight line $lx + my = 0$ from the pair of straight lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$\frac{\{l\,(hf - bg) + m\,(gh - af)\}^2}{(h^2 - ab)^{\frac{3}{2}}\,(am^2 - 2hlm + bl^2)}.$$

5. The length of the intercept cut off on $x\cos\alpha + y\sin\alpha = p$, by $ax^2 + 2bxy + cy^2 = 0$, is

$$\frac{2p\sqrt{b^2 - ac}}{a\sin^2\alpha - 2b\sin\alpha\cos\alpha + c\cos^2\alpha}.$$

6. The angles between the straight lines

$$(1 - m^2)\,y^2 + 2mxy + y\,(2am^3 - 4am) - 4am^2\,(x - a) = 0,$$

are bisected by the lines

$$y = \frac{x}{m} + am,$$

and

$$y = -mx + a\,(2m + m^3).$$

7. The equation

$$x^2 - \frac{a-b}{h}xy - y^2 = 0,$$

gives the bisectors of the angles between the lines

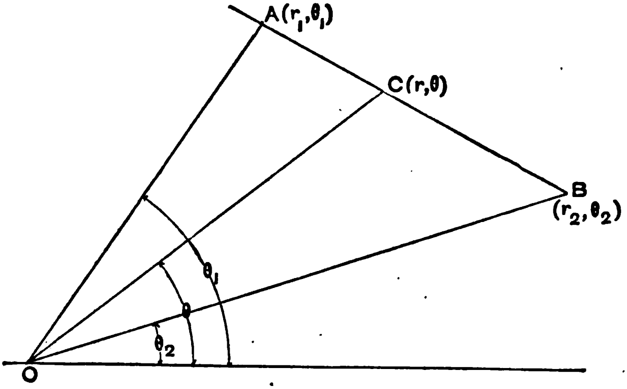
$$ax^2 + 2hxy + by^2 = 0.$$

V. THE STRAIGHT LINE REFERRED TO POLAR CO-ORDINATES.

1. The length of the perpendicular from the origin on the line joining (r_1, θ_1), (r_2, θ_2) is

$$\frac{r_1r_2 \sin(\theta_2 - \theta_1)}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}}.$$

Note. To find the equation of a straight line passing through $A\ (r_1, \theta_1)$ and $B\ (r_2, \theta_2)$. Take any point C on AB whose co-ordinates are the current co-ordinates (r, θ). Then expressing that the area of the triangle AOB equals the



sum of the triangles AOC and BOC when C lies on AB we get

$$\tfrac{1}{2}r_1r_2\sin(\theta_1-\theta_2)=\tfrac{1}{2}r_1r\sin(\theta_1-\theta)+\tfrac{1}{2}rr_2\sin(\theta-\theta_2),$$

which may be written

$$\frac{\sin(\theta_1-\theta_2)}{r}+\frac{\sin(\theta_2-\theta)}{r_1}+\frac{\sin(\theta-\theta_1)}{r_2}=0.$$

2. Find the equations of the straight lines bisecting the angles between the lines joining (r_1, θ_1), (r_2, θ_2) to the origin, and the co-ordinates of the points where they meet the line AB of the previous investigation.

3. If the bisector of the angle AOB meet AB in C, find the equation of the line joining C to the point

$$\left(\frac{2r_1r_2}{r_1+r_2}\cdot\cos\frac{\theta_2+\theta_1}{2},\ \frac{\theta_2-\theta_1}{2}\right).$$

4. Find the co-ordinates of the foot of the perpendicular dropped from the pole on the line joining (r_1, θ_1), (r_2, θ_2).

5. On OA of given length a, right-angled triangles are described of which OBA is one; the right angle being at B. On OB, P is taken such that

$$OP=\frac{\sqrt{2}\,OA^2}{OB+AB}$$

find the locus of P.

6. Two fixed straight lines intersect at C. From a given point O any straight line is drawn cutting the given lines in A and B. On AB a point P is taken so that $AP : PB :: AC : CB$; find the locus of P

7. n fixed straight lines intersect at O. From any fixed point O' any straight line is drawn to meet these lines in a, b, c... Find the locus of a point x on this straight line such that

$$\frac{n}{O'x} = \frac{1}{O'a} + \frac{1}{O'b} + \frac{1}{O'c} + \dots$$

VI. THE STRAIGHT LINE REFERRED TO TRILINEAR CO-ORDINATES.

The homogeneous form of the equation of the straight line

$$l\alpha + m\beta + n\gamma = 0$$

involves in reality only two independent constants, since we may divide out by l say, and replace the ratios $\frac{m}{l}$ and $\frac{n}{l}$ by new constants.

Let $(\alpha_1, \beta_1, \gamma_1)$ be a fixed point on a straight line. Let λ, μ, ν be the cosines of the angles which the co-ordinates $\alpha_1, \beta_1, \gamma_1$ respectively make with the line, taking care to measure in each case from the co-ordinate towards the straight line in the same direction. Now take any point (α, β, γ) on the line and at a distance r from the given point. We then obtain as another form of the equation of the straight line

$$\frac{\alpha - \alpha_1}{\lambda} = \frac{\beta - \beta_1}{\mu} = \frac{\gamma - \gamma_1}{\nu} = r.$$

The equation of a straight line passing through the intersection of $l\alpha + m\beta + n\gamma = 0$ and $l'\alpha + m'\beta + n'\gamma = 0$ may be written

$$l\alpha + m\beta + n\gamma + k\,(l'\alpha + m'\beta + n'\gamma) = 0\,;$$

for this, being of the first degree, represents some straight line, and is satisfied by the co-ordinates of the point of intersection, which require at the same time $l\alpha + m\beta + n\gamma = 0$ and $l'\alpha + m'\beta + n'\gamma = 0$. k is an indeterminate, and the straight line may therefore be made to satisfy one more condition.

To find the equation of the straight line joining the intersections of two pairs of lines expressed in the homogeneous form. Call the lines for brevity L_1, L_2, L_3, L_4, respectively, where L_1 is $l_1\alpha + m_1\beta + n_1\gamma$, then since the line required passes through the intersection of L_1 and L_2 its equation may be written

$$L_1 + kL_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly it may be written

$$L_3 + k'L_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Rendering (1) and (2) identical we have

$$\frac{l_1 + kl_2}{l_3 + k'l_4} = \frac{m_1 + km_2}{m_3 + k'm_4} = \frac{n_1 + kn_2}{n_3 + k'n_4},$$

to determine k or k', since we only require to know one of them.

A point within the triangle of reference has all its co-ordinates positive. A point that falls without the triangle, but within the angle BAC, has its α co-ordinate negative, its other two co-ordinates positive.

This example will sufficiently explain the convention with respect to the signs of co-ordinates.

Adopting it we have, if (α, β, γ) be any point in the plane of the triangle of reference,

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

where Δ is the area of the triangle of reference. When

(α, β, γ) lies without the triangle, 2Δ is the *difference* between double the sum of two of the triangles the sides subtend at the point, and double the third. When (α, β, γ) lies at infinity this finite difference becomes inappreciable when compared with the infinite values of the triangles subtended by the sides at a point at infinity.

We have then for any point at an infinite distance

$$a\alpha + b\beta + c\gamma = 0,$$

and as this is the equation of a straight line, we may assume all points at infinity to lie on a straight line.

Three straight lines will intersect in a point if the coördinates of the point of intersection of two of them satisfy the equation of the third.

Two straight lines are said to be parallel if they intersect at infinity; that is to say if the co-ordinates of their point of intersection satisfy the equation of the line at infinity.

1. Hence shew that

$$l\alpha + m\beta + n\gamma = 0, \ l'\alpha + m'\beta + n'\gamma = 0,$$

are parallel if

$$a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm) = 0.$$

2. If the perpendiculars from any point in

$$l\alpha + m\beta + n\gamma = 0,$$

on the sides of the triangle of reference make angles θ, ϕ, ψ with the straight line, then

$$l\cos\theta + m\cos\phi + n\cos\psi = 0.$$

3. The perpendicular from a point $(\alpha_1, \beta_1, \gamma_1)$ on $l\alpha + m\beta + n\gamma = 0$ is

$$\frac{l\alpha_1 + m\beta_1 + n\gamma_1}{\{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C\}^{\frac{1}{2}}}.$$

4. The two straight lines

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0,$$

are perpendicular if

$$ll' + mm' + nn' - (mn' + m'n) \cos A - (nl' + n'l) \cos B - (lm' + l'm) \cos C = 0.$$

5. Find the equations:

(i) Of the lines bisecting the angles of the triangle of reference.

(ii) Of the lines joining the vertices to the middle points of opposite sides.

(iii) Of the perpendiculars dropped from the vertices on the opposite sides.

(iv) Of the lines drawn at right angles to the sides at their middle points.

Shew that each of these sets of lines meet in a point.

6. Find the ratio of the co-ordinates of each of the points of intersection of the sets of lines in the last example.

7. Find the equations of the lines joining

(i) The points where the bisectors of the angles of the triangle meet the opposite sides.

(ii) The feet of the perpendiculars drawn from the vertices on the opposite sides.

8. Find the equation of the straight line joining the intersection of the perpendiculars of the triangle to the intersection of the bisectors of sides. Shew that this line passes through the centre of the circumscribing circle of the triangle.

9. M is a point on AB such that

$$AM = \frac{c + b\cos A}{3};$$

shew that the straight line drawn at M perpendicular to AB will meet CE, the bisector of AB, in the point of intersection of the bisectors of the sides of the triangle.

10. The equation of any straight line may be written

$$pa\alpha + qb\beta + rc\gamma = 0,$$

where p, q, r are perpendiculars on the straight line from A, B, and C respectively.

11. If p, q, r be perpendiculars from the vertices on a line passing through the intersection of the bisectors of sides; shew that, adopting a certain convention with regard to sign, $p + q + r = 0$; and secondly, if p, q, r represent perpendiculars from the vertices on the line joining the intersection of the perpendiculars from the vertices on the sides to the centre of the circumscribing circle, shew that the same relation holds between them.

12. If p, q, r be the perpendiculars from the vertices

(i) on the line joining the centres of the inscribed and circumscribed circles of the triangle of reference;

(ii) on the line joining the centre of the circumscribing circle to the intersection of the perpendiculars from the vertices on the opposite sides; find in each case the ratios $p : q : r$.

13. *AD, BE, CF* are perpendiculars from the vertices on the opposite sides of the triangle of reference; find the locus of a point such that the algebraical sum of its perpendicular distances from the sides of the triangle *DEF* is zero.

14. Shew that

$$\alpha^2+\beta^2+\gamma^2+\left(\frac{b}{c}+\frac{c}{b}\right)\beta\gamma+\left(\frac{c}{a}+\frac{a}{c}\right)\gamma\alpha+\left(\frac{a}{b}+\frac{b}{a}\right)\alpha\beta=0$$

represents only one finite straight line.

15. Two vertices *A* and *B* of a triangle *ABC* slide on fixed lines *OA, OB*. Find the locus of the vertex *C*, if *AC, BC, AB* pass through three fixed points *P, Q, R* respectively, which are situated in one and the same straight line.

16. *AD, BE, CF* are three lines through the vertices of a triangle *ABC*, intersecting in *O*. *D, E, F* are the points where they meet the sides. If *DE* and *AB* intersect in *L*, and if *M* and *N* are the other two similar intersections, shew that *L, M, N* lie on the same straight line.

17. If the co-ordinates of the point of intersection *O* in the last example be proportional to

$$l^{-1},\ m^{-1},\ n^{-1};$$

shew that

$$\begin{aligned}\sin BOC\,\{m^2+n^2-2mn\cos A\}^{-\frac{1}{2}}&\\ =\sin COA\,\{n^2+l^2-2ln\cos B\}^{-\frac{1}{2}}&\\ =\sin AOB\,\{l^2+m^2-2lm\cos C\}^{-\frac{1}{2}}&.\end{aligned}$$

18. If three straight lines join a point O to the vertices of the triangle of reference, and if p, q, r be the perpendiculars drawn from any point on OA, OB, OC respectively; shew that the algebraical sum

$$p \sin BOC + q \sin COA + r \sin AOB = 0.$$

19. If p, q, r be the perpendiculars from O on the sides DE, EF, FD in Example 16, and if P be the perpendicular from O on the line LMN, then if $(\alpha_1, \beta_1, \gamma_1)$ be the co-ordinates of O, shew that

$$\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} + \frac{9}{P^2} = 4\left(\frac{1}{\alpha_1^2} + \frac{1}{\beta_1^2} + \frac{1}{\gamma_1^2}\right).$$

20. The side BC of the triangle of reference subtends a right angle at any point (α, β, γ) whose co-ordinates satisfy the equation

$$\beta\gamma = \alpha\{\alpha \cos A - \beta \cos B - \gamma \cos C\}.$$

21. The two points at which the escribed circles of a triangle subtend equal angles lie on the straight line whose equation is

$$\alpha \cos A\,(b - c) + \beta \cos B\,(c - a) + \gamma \cos C\,(a - b) = 0.$$

22. If a straight line be drawn from the vertex of a triangle to the point where the inscribed circle touches the base, the middle point of this line, the middle point of the base, and the centre of the circle will lie in a straight line.

23. Perpendiculars are drawn from any point on the sides of the triangle of reference; find the co-ordinates of the feet of these perpendiculars.

24. A point (α, β, γ) is taken within the triangle of reference and perpendiculars are drawn on the sides, the co-ordinates of whose feet are $(\alpha_1, \beta_1, 0)$, $(0, \beta_2, \gamma_2)$, $(\alpha_3, 0, \gamma_3)$, shew that

$$\left(\frac{\alpha_1-\alpha}{\gamma}\right)^2+\left(\frac{\beta_2-\beta}{\alpha}\right)^2+\left(\frac{\gamma_3-\gamma}{\beta}\right)^2+\frac{2(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma)}{\alpha\beta\gamma}=1.$$

25. If $\beta-k_1\gamma=0$, $\gamma-k_2\alpha=0$, $\alpha-k_3\beta=0$ make angles θ, ϕ, ψ with the sides AB, BC, CA of the triangle of reference, shew that

$$\frac{\sin(A-\theta+\psi)\sin(B-\phi+\theta)\sin(C-\psi+\phi)}{\cos^2\theta\,.\,\cos^2\phi\,.\,\cos^2\psi}$$

$$=\frac{(\sin A+k_1k_2\sin B+k_2\sin C)(\sin B+k_2k_3\sin C+k_3\sin A)(\sin C+k_3k_1\sin A+k_1\sin B)}{(k_1+\cos A)^2(k_2+\cos B)^2(k_1+\cos C)^2}.$$

26. If Δ' be the area of the triangle formed by the intersection of $\beta-k_1\gamma=0$, $\gamma-k_2\alpha=0$, $\alpha-k_3\beta=0$, shew that

$$\frac{\Delta-\Delta'}{\Delta}=\frac{ak_3}{ck_2k_3+ak_3+b}+\frac{bk_1}{ak_3k_1+bk_1+c}+\frac{ck_2}{bk_1k_2+ck_2+a},$$

where Δ is the area of the triangle of reference.

VII. THE CIRCLE REFERRED TO CARTESIAN CO-ORDINATES.

1. Construct the circle whose equation is

$$x^2+y^2-2x+6y=3;$$

and determine the equation of that diameter of it which passes through the origin of co-ordinates.

2. A circle passes through the origin and through the point (3, 7); its centre is on the axis of x; find its equation.

3. Find the radii and co-ordinates of the centres of

(1) $6x^2 - 2y(7 - 3y) = 0$,

(2) $3x^2 - 6x + 3y^2 + 9y - 12 = 0$.

4. Find the radius and the co-ordinates of the centre of the circle whose equation is

$$7x^2 + 3y^2 - 4y - (1 - 2x)^2 = 0;$$

and the points where it cuts the axis of x.

5. Find the equation of the circle passing through the three points (7, 5), (– 2, 4) and (3, – 3).

6. Find the equation of the circle through the points

(7, 4), (– 5, 3) and (1, 7).

7. Shew that the equation of the circle becomes

$$x^2 + y^2 - 2ry = 0,$$

when the origin of co-ordinates is on the circumference, the centre being on the axis of y and r the radius.

8. The equation of a circle is

$$(x^2 + y^2)(1 + a^2)^{\frac{1}{2}} - 2b(x + ay) = 0;$$

find its radius.

9. The equations of the three circles are

$$x^2 + y^2 = 3 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$x^2 + y^2 - 6x - 10y + 25 = 0 \ldots\ldots\ldots(2),$$
$$x^2 + y^2 - 4(4x + y) + 52 = 0 \ldots\ldots\ldots(3).$$

Shew that the line joining the centres of (1) and (2) is perpendicular to the line joining the centres of (2) and (3).

10. Find the equation of the common chord of the two circles

$$x^2+y^2=25 \text{ and } x^2+y^2+6x-8y=0.$$

11. The equation of a circle is

$$x^2+y^2-6x-12y+41=0,$$

find its radius, and also the equation of the tangents which pass through the origin of co-ordinates.

12. Find the angle contained by those diameters of the circles

$$x^2+y^2+x+2y+1=0,$$

and $$x^2+y^2+2x+y+1=0,$$

which pass through the origin.

13. Determine the circle which has its centre at the point $(1, -3)$ and which touches the straight line

$$y-2x+4=0.$$

14. Find the points in which $y=5x+2$ meets the circle $y^2+x^2-4y-13x=9$; and the length of the part intercepted.

15. Determine the points in which $y=3x+2$ cuts the circle $y^2+x^2-4x+4y=7$.

16. Find the angle at which a tangent of the circle $x^2+y^2=16$, at a point whose abscissa is -3, cuts the axis of x.

Note. It is useful to express the fact that a point lies on the circle by writing x and y in terms of the same variable parameter. Thus in the circle

$$(x-\alpha)^2+(y-\beta)^2=r^2,$$

we may assume $x = \alpha + r\cos\theta$, $y = \beta + r\sin\theta$; to be any point on the circle by assigning different values to θ. The angle θ is included between a radius through the given point and another parallel to the axis of x. The tangent at the point θ becomes in this case

$$(x-\alpha)\cos\theta + (y-\beta)\sin\theta = r.$$

The condition that $lx + my + n = 0$ should touch the circle $(x-\alpha)^2 + (y-\beta)^2 = r^2$ is best expressed by saying that the perpendicular from the centre of the circle (α, β) equals in length the radius. This gives at once

$$\frac{l\alpha + m\beta + n}{\sqrt{l^2 + m^2}} = \pm r.$$

17. Find the conditions that the sides of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) should touch the circle $x^2 + y^2 = r^2$.

18. One of the straight lines represented by

$$a^2x^2 - b^2y^2 = 0,$$

touches the circle

$$(x-a)^2 + (y-b)^2 = r^2 \text{ if } r = \sqrt{a^2 + b^2},$$

the other if

$$r^2 = \frac{(a^2 - b^2)^2}{a^2 + b^2}.$$

19. The circle described upon the line joining the points (x_1, y_1), (x_2, y_2) as diameter, has for equation

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

20. The equation of the segment of the circle passing through (x_1, y_1), (x_2, y_2) and containing an angle α is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot\alpha \,.\, \Delta = 0,$$

where $\frac{\Delta}{2}$ is the triangle subtended by the current point (x, y) at the points (x_1, y_1), (x_2, y_2).

21. Two circles of radii a and b intersect at an angle θ; the length of the common chord is

$$\frac{2ab \sin \theta}{\sqrt{a^2 - 2ab \cos \theta + b^2}}.$$

22. The area of the triangle contained by two tangents from the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ and their chord of contact is

$$\frac{r \{x_1^2 + y_1^2 - r^2\}^{\frac{3}{2}}}{x_1^2 + y_1^2}.$$

23. Find the angle between the straight lines which join the points of intersection of $\frac{x}{a} + \frac{y}{b} = 1$ with $x^2 + y^2 = a^2$ to the origin.

24. The condition that the chord of contact of tangents from (h, k) to $x^2 + y^2 = r^2$ should subtend a right angle at (h', k') is

$$\frac{h'^2 + k'^2 - r^2}{hh' + kk' - r^2} = \frac{2r^2}{h^2 + k^2}.$$

25. If x_1, y_1 be the co-ordinates of the middle point of the chord of a circle, the equation of the chord may be written

$$xx_1 + yy_1 = x_1^2 + y_1^2.$$

26. The locus of the middle points of chords of a circle passing through a given point is a circle passing through the centre, and described on the distance between it and the given point as diameter.

27. Two circles touch; shew that the perpendicular distance of any point from their common tangent at the point of contact varies as the difference of the squares of the tangents drawn from it to the two circles.

28. Find the equation of the circle which is inscribed in the triangle formed by the two tangents to the circle $x^2+y^2=a^2$ from the point $(2a, 0)$ and their chord of contact.

29. Find the angle contained by either tangent from (h, k) to the circle $(x-\alpha)^2+(y-\beta)^2=r^2$ and the polar of the point with respect to the same circle.

30. Find the locus of a point the chord of contact of tangents from which to a circle is of constant length. See Note after (34).

31. Find the condition that the polars of (h, k) and (h', k') with respect to the circle $x^2+y^2=r^2$ should intersect on the circumference.

32. If the arc intercepted on

$$x^2+y^2=2ax \text{ by } x^2+y^2\pm 2by=0,$$

and external to both circles is equal to either of the arcs intercepted by the same circles and external to both on $x^2+y^2=b^2$; shew that $ab\sin\left(\frac{\pi b}{6a}\right)$ is a harmonic mean between a^2 and b^2.

33. If through the middle point of any chord of a circle two chords are drawn, the straight lines joining their extremities will intersect the first chord at equal distances from the middle point.

34. The tangent at any point of a circle bisects the intercepts made upon the tangents at the extremities of a diameter by the lines joining the point to the extremities of the diameter.

Note. The equation of the chord joining two points $(a\cos\theta, a\sin\theta)$ $(a\cos\phi, a\sin\phi)$ which lie on the circle $x^2+y^2=a^2$, can be shewn to be

$$x\cos\frac{\theta+\phi}{2}+y\sin\frac{\theta+\phi}{2}=a\cos\frac{\theta-\phi}{2}\,\ldots\ldots\ldots(1).$$

Let (x_1, y_1) be the pole of this chord with respect to the circle; then (1) may be written

$$xx_1+yy_1=a^2\,\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Comparing (1) and (2) we find therefore that the tangents at any two points

$$(a\cos 0, a\sin\theta), (a\cos\phi, a\sin\phi)$$

intersect at the point whose coordinates are

$$x_1=a\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\quad y_1=a\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

35. Three tangents are drawn to the circle $x^2+y^2=a^2$, two of the vertices of the triangle formed by them lie on $y=mx+c$, $y=m'x+c'$, shew that the third vertex will lie on the line

$$(mc'-m'c)\,y+(c'-c)\,x+a\,(m'-m)=0,$$

if

$$\frac{a^2}{cc'}=\frac{1}{1+mm'}.$$

36. Find the equation of the circle cutting orthogonally the circles

$$x^2+y^2=a^2;$$

$$(x-b)^2+y^2=a^2;$$

$$x^2+(y-c)^2=a^2.$$

37. Find the locus of a point the lengths of whose tangents drawn to two fixed circles have a constant ratio.

38. If the co-ordinates of A, B, C, three points on a circle, be α, β, γ in the eccentric notation and the tangent at C meet AB in D, shew that

$$CD=\frac{2R}{\cot\frac{\beta-\gamma}{2}+\cot\frac{\alpha-\gamma}{2}}$$

where R is the radius of the circle.

39. A, B, C are three points in the plane of a circle whose centre is O and radius r. Shew that the area of the triangle formed by the intersections of the polars of A, B, C, with respect to the circle is

$$\frac{r^4(ABC)^2}{4(AOB).(BOC).(COA)},$$

(ABC) &c. denoting the area of the triangle formed by the points A, B, C, &c.

40. ABC is a triangle, AA' the perpendicular on BC. Taking $A'C$ and $A'A$ as axes of x and y, the equation of the nine-point circle of the triangle is

$$x^2+y^2-R\{x\sin(B-C)+y\cos(B-C)\}=0,$$

R being the radius of the circumscribing circle.

41. Find the co-ordinates of the centre and the radius of the circle in (40).

42. Find the distance between the centres of the circumscribing and the nine-point circle of the triangle ABC.

43. If t_1, t_2, t_3 be the lengths of tangents that are drawn from the vertices of the triangle ABC to its nine-point circle, shew that with the notation of (39)

$$t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2 = (ABC)^2.$$

44. If l, m, n be the lengths of the chords intercepted by the nine-point circle on the sides a, b, c of the triangle ABC respectively, shew that the algebraical sum

$$al + bm + cn$$

is zero.

45. Find the equation of the circle having the point of intersection of the perpendiculars from the vertices of a triangle on the opposite sides as centre; and the product of the segments into which this intersection divides any of the perpendiculars equal to the square of its radius.

46. If t_1, t_2, t_3 be the lengths of the tangents drawn from the vertices to the polar circle of the triangle and if R be the radius of the circumscribing circle, shew that

$$t_1^2 + t_2^2 + t_3^2 = 4R^2 (1 - 5 \cos A \cos B \cos C).$$

47. Find the equation of the circle circumscribing the triangle formed by the three lines

$$y = m_1 x + \frac{a}{m_1}, \quad y = m_2 x + \frac{a}{m_2}, \quad y = m_3 x + \frac{a}{m_3},$$

and shew that it passes through the point

$$x = a, \ y = 0.$$

48. Find the equation of a circle referred to two fixed tangents as axes, that are each of length b and include an angle ω.

49. A point P is taken on a circle whose distance PM from a given chord is equal to half that chord. Prove that the distance of the middle point of PM from the centre of the circle is equal to the sum of one quarter of the chord, and its distance from the centre.

50. A straight line slides on two lines at right angles and always touches a circle of radius r and having the intersection of the lines as centre. If l be the length of line intercepted by the two given lines shew that the intercepts on them are given by the roots of the quadratic

$$x^2 - \sqrt{(l^2 + 2lr)} \, . \, x + lr = 0.$$

51. On the tangents to a circle constant lengths are measured on each from their points of contact. Find the locus of the extremities.

52. The line joining the feet of the perpendiculars dropped from any point of the circumference of a circle on two fixed diameters is of constant length.

53. AP, BQ are chords at the extremities of a diameter AB and intersect in R. If d be the diameter of the circle, shew that

$$AP \, . \, RA + BQ \, . \, RB = d^2.$$

54. Two circles touch at C; on any chord through C meeting the circumferences in B and A a point P is

taken, and PM is the perpendicular from P on the common tangent. Shew that

$$PC = \frac{2PM}{AB}\cdot(a+b)\ ;$$

where a and b are the radii of the circles.

VIII. THE CIRCLE REFERRED TO POLAR CO-ORDINATES.

Taking the pole at a distance c from the centre of a circle of radius a, the equation of the circle is

$$r^2 - 2cr\cos\theta + c^2 = a^2.$$

When the pole lies on the circumference this reduces to $r = 2a\cos\theta$, by putting $c = a$; when the pole is the centre $c = 0$, and the equation of the circle is $r = a$.

The equation of the chord joining two points (a, θ_1), (a, θ_2), on the circle having the centre as pole may be found directly, or by substituting $r_1 = a$, $r_2 = a$ in the equation

$$\frac{\sin(\theta_2 - \theta_1)}{r} + \frac{\sin(\theta - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta)}{r_2} = 0,$$

to be

$$\frac{a}{r} = \frac{\cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right)}{\cos\frac{\theta_2 - \theta_1}{2}}\ ;$$

hence the tangent at a point α is

$$\frac{a}{r} = \cos(\theta - \alpha).$$

1. The equation of the chord joining two points α, β on the circle $r = 2a\cos\theta$ is

$$r\cos(\alpha+\beta-\theta) = 2a\cos\alpha\cos\beta.$$

2. If a_1, a_2 are the radii of the two circles which touch the circle in (1) at the origin and also touch the given chord; shew that

$$\frac{a}{a_1} + \frac{a}{a_2} = \sec\alpha \,.\, \sec\beta \,.\, \cos(\alpha+\beta).$$

3. Two diameters of a circle intersect at right angles. Taking one of them as initial line and the centre of the circle as pole, shew that the middle points of the intercepts made by the diameters on tangents to the circle lie on the curve

$$\frac{a}{r} = \sin 2\theta.$$

4. From a given point O a chord is drawn to cut a given circle in P and Q, and R is taken on this chord so that

$$\frac{1}{OR^2} = OP^2 + OQ^2 + OP \,.\, OQ,$$

find the locus of R.

5. The equation of the circle on the line joining (r_1, θ_1), (r_2, θ_2) as diameter is

$$r^2 = rr_1\cos(\theta-\theta_1) - r_1r_2\cos(\theta_1-\theta_2) + r_2r\cos(\theta_2-\theta).$$

6. The equation of the tangent to a circle whose centre is (c, α) at a point (r_1, θ_1), is

$$r_1^2 = r_1c\cos(\theta_1-\alpha) - cr\cos(\alpha-\theta) + rr_1\cos(\theta-\theta_1).$$

7. If in the last example the point of contact be P and PQ is measured along the tangent from P, of constant length l; shew that the locus of Q is

$$r^2 - 2cr\cos(\theta - \alpha) = a^2 + l^2 - c^2,$$

a being the radius of the circle.

8. The vertex C of a triangle is fixed. The length of the base is $2c$ and the sides are of the form $2c\lambda$, $2c\sqrt{1-\lambda^2}$, where λ is variable. Find the locus of the middle point of the base.

IX. THE RADICAL AXIS OF TWO OR MORE CIRCLES.

If $S=0$ and $S'=0$ denote two circles, $S-kS'=0$ is the equation of a circle passing through their points of intersection, real or imaginary. If the coefficients of x^2 and y^2 in S and S' are unity, then when k has the value unity it will be seen that $S-S'=0$ is the equation of a straight line passing through the real or imaginary points of intersection, called the radical axis of the two circles. Thus any two circles have a radical axis whether they meet or not. It is best to take the radical axis and line of centres of the circles as axes of y and of x respectively, when a system of circles having the same radical axis may be written

$$x^2 + y^2 - 2hx \pm b^2 = 0,$$

where b is the same for all circles of the system and h varies, being the distance of the centre of each from the radical axis. The points where the axis of y meets the circles are found (putting $x=0$) to be $y^2 \pm b^2 = 0$, which are real when b^2 is negative, that is when r, the radius of the circle, is greater than h, and imaginary when b^2 is positive, i.e. when r is less than h.

1. The square of the tangent that can be drawn from any point on one circle to another is proportional to the perpendicular from the point on their radical axis.

2. If two circles touch, their common tangents subtend right angles at the point of contact.

3. Find the equation of the system of circles cutting the system $x^2+y^2-2hx=b^2$ orthogonally.

4. Two circles touch. A chord of one circle touches the other, find the locus of its middle point.

5. Shew that the common tangents to two circles intersect the line joining their centres in points, the ratios of whose distances from the two centres is equal to the ratio of the radii of the circles.

X. THE ELLIPSE REFERRED TO THE CENTRE AS ORIGIN.

[*Unless otherwise mentioned,* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is taken as the equation of the ellipse.*]

1. Shew that the locus of a point whose distances from two fixed points are together equal to a given line is an ellipse whose major axis is the length of that line.

2. If a straight line of given length move so that its extremities are always on two straight lines at right angles to one another; shew that any point in it will describe an ellipse, whose semi-axes are the segments of the line. Explain the case in which the point bisects the line.

3. A tangent to an ellipse is inclined at an angle of 45° to the axes; find the co-ordinates of its point of contact with the curve.

4. Find the point in which the line joining a focus to an extremity of the minor axis meets the curve again.

5. The locus of the point whose distances from a given point in the axis of x and from a given straight line parallel to the axis of y are in the ratio $e : 1$ is the curve

$$y^2 + (1 - e^2)\,x^2 - 2m\,(1 \pm e)\,x = 0,$$

m being the distance of the given point from the origin, and the line being at such a distance from the axis of y that the locus passes through the origin.

Note. It is useful to denote a point on the ellipse by $(a \cos \theta, b \sin \theta)$ in the eccentric notation. By referring to a point θ on the ellipse the point $(a \cos \theta, b \sin \theta)$ will be meant. Let C be the centre of the ellipse, then if PN be drawn perpendicular to the major axis and produced to meet the auxiliary circle in Q, we see that

$$CN = a \cos \theta = CQ \cos QCN,$$

$$\therefore \angle \theta = \angle QCN.$$

6. The equation of the chord joining two points θ and ϕ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is
$$\frac{x}{a} \cos \frac{\theta + \phi}{2} + \frac{y}{b} \sin \frac{\theta + \phi}{2} = \cos \frac{\theta - \phi}{2}.$$

Hence deduce the equation of the tangent at θ.

7. Find the length of the chord joining the points θ and ϕ on the ellipse.

8. Given the co-ordinates of a point at the extremity of a diameter firstly in the form (x_1, y_1), secondly in the form $(a\cos\theta, b\sin\theta)$, find the co-ordinates of the extremities of the diameter conjugate to the same. Hence shew that the sum of the squares of two conjugate semi-diameters is constant and equal to a^2+b^2.

9. Find the co-ordinates of the points of intersection of the tangents at the points θ and ϕ.

10. If (x_1, y_1) be the co-ordinates of the middle point of a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, shew that the equation of the chord may be written

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

11. Through a fixed point straight lines are drawn to cut the ellipse. Find the locus of the middle points of the chords which they intercept.

12. A triangle is inscribed in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the co-ordinates of the middle points of its sides, shew that they satisfy

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}\right)=\frac{(x_2y_1-x_1y_2)^2}{a^2(y_2-y_1)^2+b^2(x_2-x_1)^2}$$

and two similar equations.

13. The length of a chord of the ellipse in terms of (x_1, y_1), its middle point, is

$$2ab\left\{\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)^{\frac{1}{2}}\left[\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^{-1}-1\right]\right\}^{\frac{1}{2}}.$$

14. In general one chord of an ellipse and only one is bisected at right angles by $\frac{x}{p}+\frac{y}{q}=1$.

Find the co-ordinates of the point of bisection.

15. If a circle be described on a chord of an ellipse as diameter whose middle point is (x_1, y_1), and if (x_2, y_2) be the middle point of the chord joining the two remaining points of intersection of the curves, shew that

$$\frac{y_2-y_1}{a^2y_2}=\frac{x_2-x_1}{b^2x_2}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$x_1y_2+y_1x_2=0\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

16. Shew that the equation of the second chord in (15) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)\left(\frac{a^2+b^2}{a^2-b^2}\right)=0.$$

17. The condition that the polars of (x_1, y_1), (x_2, y_2) should intersect on the ellipse is

$$a^2(y_2-y_1)^2+b^2(x_2-x_1)^2=(x_2y_1-x_1y_2)^2.$$

18. From a point (x_1, y_1) tangents are drawn to the ellipse. The area of the triangle formed by the tangents and their chord of contact is

$$\frac{ab\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)^{\frac{3}{2}}}{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}.$$

Note. To find the angle between the tangents that can be drawn to the ellipse from a given point (x_1, y_1). Using the eccentric notation, if a tangent at the point θ passes through (x_1, y_1) we have

$$\frac{x_1}{a}\cos\theta + \frac{y_1}{b}\sin\theta = 1,$$

or substituting for $\cos\theta$ and $\sin\theta$ in terms of $\cot\theta$,

$$\left(1 - \frac{x_1^2}{a^2}\right)\cot^2\theta - \frac{2x_1y_1}{ab}\cot\theta + 1 - \frac{y_1^2}{b^2} = 0 \text{.........}(1).$$

If $\cot\theta_1$ and $\cot\theta_2$ are the roots of this quadratic, m_1 and m_2 the tangents of the angles which the tangents make with the axis of x, and ϕ the angle they include; we have

$$m_1 = -\frac{b}{a}\cot\theta_1,\ m_2 = -\frac{b}{a}\cot\theta_2,$$

and

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2ab(\cot\theta_2 - \cot\theta_1)}{a^2 + b^2\cot\theta_2\cot\theta_1} \text{..........}(2).$$

Substituting from (1) in (2) we obtain

$$\tan\phi = \frac{2ab\sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1}}{x^2 + y^2 - a^2 - b^2}.$$

19. The locus of tangents cutting at right angles is a circle concentric with the given ellipse.

20. The tangents of the angles which the tangents to the ellipse from the point (x_1, y_1) make with the axis of x are given by the quadratic

$$(a^2 - x_1^2)m^2 + 2x_1y_1m + (b^2 - y_1^2) = 0.$$

21. Find the lengths intercepted by the tangents from (x_1, y_1) on the major and minor axes respectively.

22. If (h, k) be the points of intersection of the tangents at the points (x_1, y_1), (x_2, y_2); shew that the normals will intersect in the point

$$x = \frac{a^2 - b^2}{a^4} \cdot x_1 x_2 h, \quad y = -\frac{a^2 - b^2}{b^4} \cdot y_1 y_2 k.$$

23. If (h, k) and (α, β) be the intersection of the tangents and normals respectively at the extremities of a pair of conjugate diameters, shew that

$$\frac{\alpha k}{\beta h} = \frac{b^2}{a^2}.$$

24. The tangents from (x_1, y_1) to the ellipse intersect at right angles. Shew that the intersection of the normals at the points of contact lie on the straight line

$$\frac{y}{x} = \frac{y_1}{x_1}.$$

25. If the line $\frac{y}{x} = \frac{y_1}{x_1}$ in the last example meet the polar of (x_1, y_1) in (h, k) shew that

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = \sqrt{\frac{h^2 + k^2}{a^2 + b^2}}.$$

26. The chord joining the points α, β on the ellipse will subtend a right angle at a vertex if

$$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} + \frac{a^2}{b^2} = 0;$$

at the centre if

$$\tan\alpha \cdot \tan\beta + \frac{a^2}{b^2} = 0.$$

27. Find the locus of the pole of a chord of an ellipse when it subtends a right angle

(1) at a vertex,

(2) at the centre.

28. Find the locus of the pole of the chord the perpendicular from the centre on which is constant.

29. The length of the polar chord of the point (x_1, y_1) is

$$\frac{2ab\sqrt{S}}{p\,(S+1)},$$

where p is the length of the perpendicular on it from the centre and

$$S=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1.$$

30. Find the equation of the circle passing through the lower extremity of the minor axis and the foci.

Find where it meets the minor axis again.

31. If from the lower extremity B' of the minor axis $B'B$ a length $B'D$ be measured along it, such that it, the semi-major, and semi-minor axes form a continued proportion, shew that $B'D$ will subtend right angles at the foci and at the points whose ordinates are $\frac{b^3}{a^2-b^2}$.

32. If from a point T whose co-ordinates are x and y tangents TP, TQ are drawn to an ellipse whose centre is C and equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

shew that

$$CP\cos PCT + CQ\cos QCT = \frac{2CT}{\frac{x^2}{a^2}+\frac{y^2}{b^2}}.$$

33. A chord of an ellipse moves parallel to itself; shew that the sum of the eccentric angles at its extremities is constant.

34. The tangents to the ellipse $bx^2 + ay^2 = ab(a+b)$ at the points where it is cut by $bx\cos\theta + ay\sin\theta = ab$ are at right angles to one another.

35. If a common tangent be drawn to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{c}=0, \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}+\frac{2x}{c}=0,$$

it will subtend a right angle at the origin.

36. Find the locus of the point of intersection, other than the centre of the ellipse, of the circles described on two conjugate semi-diameters as diameters.

37. Find the locus of the middle point of the chord joining the extremities of two conjugate diameters.

38. Find the locus of intersection of the bisectors of the sides from opposite vertices in the triangle formed in a given ellipse by two conjugate semi-diameters and the chord joining their extremities.

39. On PQ, a chord joining the extremities of two conjugate diameters, an equilateral triangle is described. Find the locus of the vertex.

40. The centre of the circle circumscribing the triangle formed by two conjugate semi-diameters and their chord is found by the intersection of the straight lines

$$\left.\begin{aligned} by\sin\alpha + ax\cos\alpha &= \frac{b^2\sin^2\alpha + a^2\cos^2\alpha}{2} \\ by\cos\alpha - ax\sin\alpha &= \frac{b^2\cos^2\alpha + a^2\sin^2\alpha}{2} \end{aligned}\right\},$$

where α is the co-ordinate of the extremity of one of the diameters.

41. The chord joining the extremity of the diameter which passes through the point α to the extremity of its conjugate in the ellipse

$$a^2x^2 + b^2y^2 = \frac{(a^2+b^2)^2}{8}$$

passes through the centre of the circle circumscribing the triangle formed by the semi-diameter through the point α, its conjugate, and the chord joining their extremities in the ellipse $a^2x^2 + b^2y^2 = a^2b^2$.

42. The chord joining (x_1, y_1) a point on the ellipse to the extremity of the diameter conjugate to that through (x_1, y_1) subtends right angles on the axis of x at points given by

$$abx^2 - ax(bx_1 - ay_1) = (a^2 - b^2)\,x_1y_1.$$

43. Four points α, β, γ, δ on the ellipse will be circumscribable by a circle if $\alpha+\beta+\gamma+\delta$ is an even multiple of two right angles.

PROPERTIES OF THE NORMAL.

44. Find the equation of the normal at the point

$$(a \cos \theta, b \sin \theta).$$

45. The normal at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will always pass through the point which has the same eccentric angle on $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$, if $aa'-bb'=a^2-b^2$.

46. If $aa'=bb'$, the locus of the intersection of normals to the two ellipses at points having the same eccentric angle is the ellipse

$$\frac{x^2}{(a+a')^2}+\frac{y^2}{(b+b')^2}=1.$$

47. The normals at the points whose eccentric angles are α, β, γ will meet in a point if

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$$

48. The normal at (h, k) will meet the ellipse again in the point

$$\frac{x}{h(a^2-b^2-p^2)}=\frac{y}{k(-a^2+b^2-p^2)}=\frac{1}{a^2+b^2-p^2},$$

p being the perpendicular from the centre on the tangent at (h, k).

49. If α, β, γ, δ be the eccentric angles of the points where the normals from a given point meet the ellipse, shew that $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.

50. The tangents of the inclinations to the axis of x of the normals from the point (x, y) to the ellipse are given by the equation

$$b^2x^2m^4 - 2b^2xym^3 + m^2\{a^2x^2 + b^2y^2 - (a^2 - b^2)^2\} - 2a^2xym + a^2y^2 = 0.$$

51. Find the locus of the points such that if normals be drawn from them to the ellipse, the sum of their inclinations to the axis of x is constant and equal to ϕ.

52. On the normal at P a length PQ is taken such that $PQ^2 = CQ^2 + CD^2$, C being the centre and CD the diameter conjugate to CP. Shew that the locus of Q is the concentric ellipse

$$x^2 + (1 - e^2)\, y^2 = \frac{a^2e^4}{4}.$$

53. The normal at P meets the diameter conjugate to the one at P in Q; shew that the co-ordinates of Q satisfy the relation

$$(x^2 + y^2)\sqrt{a^2y^2 + b^2x^2} = (a^2 - b^2)\, xy.$$

54. In the last example if C be the centre, CD the semi-diameter conjugate to CP, and if (x_1, y_1) be the co-ordinates of P, shew that

$$CQ \,.\, CD = \left(\frac{a}{b} - \frac{b}{a}\right) x_1y_1,$$

and that

$$\tan CPQ = \left(\frac{1}{b^2} - \frac{1}{a^2}\right) x_1y_1.$$

55. Find the locus of the points of intersection of the normals at the extremities of pairs of conjugate diameters.

56. The equation of the ellipse referred to the tangent and normal at any point as axes of x and y respectively, at which a' is the diameter, b' its conjugate, ω the angle between the tangent at the point and the diameter a', is

$$\frac{x^2}{b'^2}+y^2\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{b'^2}\right)-\frac{2xy\cot\omega}{b'^2}-\frac{2yb'}{ab}=0.$$

57. Find the length of normal intercepted at a point P, whose diameter is a' and its conjugate b'. If CD be the semi-diameter conjugate to CP, find the length intercepted on the normal at D.

58. If N and N' be the lengths of normal chords intercepted by the ellipse at points P and D which are the extremities of two conjugate semi-diameters of lengths a' and b' respectively, shew that $\frac{b'^2}{N}$ and $\frac{a'^2}{N'}$ are the roots of the quadratic

$$x^2-\frac{a^4+b^4}{2ab}x+\frac{(a^2+b^2)^2\cot^2\omega+a^2b^2}{4}=0,$$

ω being the angle PCD.

PROPERTIES OF THE FOCI.

59. The semi-ordinate MP of a point P on an ellipse is produced to meet the auxiliary circle in Q. If C be the centre of the ellipse, S a focus and e the eccentricity, shew that

$$\tan\tfrac{1}{2}PSC=\sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}QCM.$$

60. In the last example find the locus of intersection of SP and CQ.

61. If tangents be drawn to an ellipse from any point P, and if T and T' are the points of contact, shew that PT and PT' subtend equal angles at either focus.

62. If p be the length of the perpendicular from the focus S on a tangent at P, shew that

$$\frac{b^2}{p^2} = \frac{2a}{SP} - 1,$$

b and a being the semi-minor and semi-major axes.

63. If l and l' be the lengths of the tangents from a given point at a distance d from the focus S, if r and r' be the radii vectores from S to their points of contact, and p and p' the perpendiculars from S on the tangents, shew that

$$(1)\quad \frac{l^2r' - l'^2r}{r' - r} = d^2 - rr',$$

$$(2)\quad \frac{lp' - l'p}{l'p' - lp} = \frac{d^2 - rr'}{ll'}.$$

64. P, $(a\cos\theta,\ b\sin\theta)$ is joined to the foci S and S'. SP and $S'P$ meet the curve again in Q and R. If Q and R have ϕ_1 and ϕ_2 as their eccentric angles, shew that ϕ_1 and ϕ_2 are determined from the relation

$$\tan\frac{\phi}{2} = \frac{\pm e - 1}{1 \pm e} \,.\, \cot\frac{\theta}{2}.$$

65. The equation of QR in the last example is

$$-\frac{x}{a}\cos\theta + \frac{y}{b} \,.\, \frac{e^2 + 1}{e^2 - 1} \,.\, \sin\theta = 1.$$

66. The focal radii of a point P meet the curve again in Q and R. Shew that the pole of QR will lie on the normal at P.

67. SM and $S'M$ are drawn perpendicular to the focal distances of $P(x_1, y_1)$, viz. SP and $S'P$. Find the coordinates of M.

68. The circle described on the focal distance of any point touches the auxiliary circle.

69. If p and p' be the perpendiculars from the centre C and the focus S on the tangent at a point P such that $SP = r$, shew that

$$p'a = pr,$$

a being the major semi-axis; and if a parallel to the tangent through C meet SP in E and $S'P$ in E', shew that

$$EP = E'P = a.$$

70. ABC is a triangle of constant base BC. The perpendicular at O, the middle point of BC, meets the sides or the sides produced in M and N; find the locus of A if $OM \,.\, ON = b^2$.

71. A triangle is inscribed in an ellipse. The lines drawn from the vertices to bisect the opposite sides meet them in (x_1, y_1), (x_2, y_2), (x_3, y_3) and intersect at a focus; shew that

$$\frac{x_1^2 + x_2^2 + x_3^2}{a^2} + \frac{y_1^2 + y_2^2 + y_3^2}{b^2} = \frac{28a^2 - b^2}{36a^2}.$$

72. The circle through the foci and any point of an ellipse will pass through the intersection of the minor axis and the normal at the point.

Note. When a system of conics has the same foci the conics are said to be confocal.

If λ denote the half distance between the foci, any conic of the system whose semi-axis is a may be written

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-\lambda^2}=1.$$

Again, we may write the equation of any conic of a confocal system in the form

$$\frac{x^2}{a^2\pm k}+\frac{y^2}{b^2\pm k}=1$$

if $\sqrt{a^2-b^2}$ is the half distance between the foci; where by giving k different values, the various conics of the system are obtained.

73. Confocal ellipses intersect at right angles.

74. In general two conics of a confocal system can be drawn through any point in the plane, of which one is an ellipse, the other a hyperbola.

75. Find the locus of the pole of a straight line $y=mx+c$ with respect to a system of confocal ellipses.

76. Two straight lines cut at right angles; they are tangents respectively to two confocal conics; find the locus of their intersection.

77. From a point in the axis of x, tangents are drawn to a system of confocal conics; find the locus of the points of contact.

78. Two ellipses are confocal. A tangent is drawn to the one at a point P, and Q is the pole of this tangent with respect to the other ellipse. If p be the perpendicular from the centre on this tangent, shew that

$$PQ=\frac{a_2^2-a_1^2}{p};$$

where $2a_2$ and $2a_1$ are the major axes of the two ellipses.

XI. THE ELLIPSE REFERRED TO POLAR CO-ORDINATES, THE FOCUS BEING THE POLE.

If l be half the latus rectum, the equation of the conic is

$$\frac{l}{r} = 1 + e \cos \theta,$$

which is an ellipse or a hyperbola according as e is less or greater than unity.

Note I. To find the equation of a chord of this conic. Take $\alpha + \beta$, $\alpha - \beta$, as the angular co-ordinates of the two points it is to join; and assume it of the form

$$\frac{l}{r} = A \cos \theta + B \cos (\theta - \alpha)$$

where A and B are constants to be determined.

Then, since the chord passes through $\alpha + \beta$, we have

$$1 + e \cos (\alpha + \beta) = A \cos (\alpha + \beta) + B \cos \beta \ldots\ldots (1);$$

and since it passes through $\alpha - \beta$, we have

$$1 + e \cos (\alpha - \beta) = A \cos (\alpha - \beta) + B \cos \beta \ldots\ldots (2);$$

whence $A = e$, $B = \sec \beta$, and the chord becomes

$$\frac{l}{r} = e \cos \theta + \sec \beta \cos (\theta - \alpha).$$

When $\alpha + \beta$ and $\alpha - \beta$ coincide $\beta = 0$, and we infer the equation of the tangent at the point α to be

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha).$$

Note II. Since the tangents from any point to an ellipse subtend equal angles at a focus, it follows that the angular co-ordinate of the intersection of tangents at points whose angular co-ordinates are α and β is $\frac{\alpha+\beta}{2}$.

1. Find the locus of a point, from which if tangents be drawn, their points of contact will subtend a constant angle 2α at the focus.

2. Find the locus of the point where the chord PQ subtending a constant angle 2α at the focus is met by the internal bisector of the angle PSQ.

3. The normal to the ellipse $\frac{l}{r} = 1 + e\cos\theta$ at the point α is given by

$$\frac{l}{r} = \frac{1+e\cos\alpha}{e}\cdot\left\{\frac{e\sin\theta + \sin(\theta-\alpha)}{\sin\alpha}\right\}.$$

4. If β be the angular co-ordinate of the point where the normal at α meets the curve again, shew that

$$\tan\frac{\beta}{2}\cdot\tan\frac{\alpha}{2} = -\frac{e^2+e\cos\alpha+1+e}{e^2+e\cos\alpha+1-e}.$$

5. The polar of the point (c, α) with respect to

$$\frac{l}{r} = 1 + e\cos\theta$$

is
$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{c} - e\cos\alpha\right) = \cos(\theta-\alpha).$$

6. Two conics having a common focus, latera recta l_1, l_2, eccentricities e_1, e_2, and their axes inclined to each other at an angle α, will touch if

$$(l_2 - l_1)^2 = e_1^2 l_2^2 + e_2^2 l_1^2 - 2e_1e_2l_1l_2\cos\alpha.$$

7. The focus of an ellipse, of eccentricity e, and latus rectum $2l$, is the centre of a circle of radius a. The point of contact on the ellipse of the common tangent to the circle and ellipse is given by

$$\alpha = \cos^{-1} \frac{\frac{l^2}{a^2} - (1 + e^2)}{2e}.$$

8. A tangent is drawn to an ellipse at Q, and the perpendicular from the focus S meets the curve in P and the tangent in P'; shew that

$$\frac{1}{SP} - \frac{1}{SP'} = \frac{1}{l}\left\{1 - \frac{SP'}{SQ}\right\}.$$

9. A circle touches a conic $\frac{l}{r} = 1 + e\cos\theta$ at the point $\theta = \alpha$ and passes through the pole. Shew that the equation of the chord which joins the other two points of intersection is

$$\frac{l}{r} \cdot \frac{1 + 2e\cos\alpha + e^2}{e^2} = e\cos\theta + \cos(\theta + \alpha).$$

10. On each radius vector SP a length SQ is taken equal to the length of the perpendicular from the focus on the tangent at P. Find the locus of Q.

11. From P tangents PQ and PQ' are drawn to an ellipse, SP meets the curve in R; shew that the angle

$$QSQ' = 4\cos^{-1}\left\{\frac{\frac{1}{SQ} + \frac{1}{SQ'} + \frac{2}{SP}}{\frac{4}{SR}}\right\}^{\frac{1}{2}}.$$

12. The angle between the radius vector and the tangent at a point α is $\tan^{-1} \frac{1 + e\cos\alpha}{e\sin\alpha}$.

13. If the tangent and normal at P meet the major axis in Q and R, shew that $\frac{1}{SQ} - \frac{1}{SR}$ is constant for any position of P.

14. If in the previous example PQ and PR, produced if necessary, meet the *latus rectum* through the pole in q and r respectively; shew that

$$\frac{l - SP}{Sr} + \frac{l}{Sq} = e \tan SPQ.$$

15. If a circle be described on the focal radius vector to the point on the ellipse, whose angular co-ordinate is α, as diameter, shew that the remaining points of intersection of the curves are given by

$$\tan^3 \frac{x}{2} + \tan^2 \frac{x}{2} . \tan \frac{\alpha}{2} (2e - 1) + \tan \frac{x}{2} (2e + 1) = \tan \frac{\alpha}{2}.$$

16. If α_1, α_2, α_3 be the angular co-ordinates of the points where the circle in (15) meets the ellipse again, shew that

$$\alpha_1 + \alpha_2 + \alpha_3 - \alpha$$

is an even multiple of two right angles.

17. Two conics of latera recta l_1, l_2, and eccentricities e_1, e_2, have the same focus, but their axes at right angles. If any chord through the focus S meet the first conic in P and P', the second in Q and Q', shew that

$$(1) \quad \frac{e_2^2 l_1^2}{SP . SP'} + \frac{e_1^2 l_2^2}{SQ . SQ'} + (1 - e_1^2)(1 - e_2^2) = 1.$$

$$(2) \quad \frac{1}{SP} \cdot \frac{1}{SQ'} - \frac{1}{SQ} \cdot \frac{1}{SP'} = \frac{1}{l_2}\left(\frac{1}{SP} - \frac{1}{SP'}\right) + \frac{1}{l_1}\left(\frac{1}{SQ'} - \frac{1}{SQ}\right).$$

18. The conics

$$\frac{l}{r} = 1 + e\cos\theta,$$

$$\frac{l'}{r} = 1 + e'\cos\theta,$$

cut at a point whose angular co-ordinate is

$$\cos^{-1}\left\{\frac{l'-l}{e'l-el'}\right\},$$

and at an angle

$$\tan^{-1}\left\{\frac{(e'-e)[\{(e'-1)l-(e-1)l'\}\{(e'+1)l-(e+1)l'\}]^{\frac{1}{2}}}{e'l'-el+ee'(e'l-el')}\right\}.$$

19. *P* is any point on the ellipse. The bisector of the angle *PSS'* meets *PS'* in *G*; find the locus of *G*.

20. *S* and *S'* being the foci and *P* any point on the ellipse, find the locus of the centre of the circle inscribed in the triangle *PSS'*.

XII. MISCELLANEOUS EXAMPLES ON THE ELLIPSE.

1. Tangents are drawn from a point *P* to an ellipse; find the locus of *P* if $x_1x_2 + y_1y_2 = 0$; (x_1, y_1), (x_2, y_2) being the points of contact of the tangents.

2. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angles are θ, $\theta + \frac{2\pi}{3}$, $\theta + \frac{4\pi}{3}$, shew that

(1) $x_1 + x_2 + x_3 = 0$.

(2) $y_1 + y_2 + y_3 = 0$.

(3) The sum of the squares of the distances of the three points from the origin is constant for all values of θ.

3. Find the locus of a point from which equal tangents can be drawn to a central conic.

4. Find the locus of intersections of tangents at the extremities of pairs of conjugate diameters.

5. Three tangents are drawn to an ellipse. Two vertices of the triangle, which their intersections form, slide on the fixed lines $y = mx + c$, $y = m'x + c'$ respectively; shew that the third vertex will slide on the line

$$\frac{x}{a^2}(c' - c) + \frac{y}{b^2}(c'm - cm') + m' - m = 0,$$

provided that

$$a^2mm' + b^2 = cc'.$$

6. The point of intersection, other than the centre of the ellipse, of the circles described on two semi-diameters at right angles as diameters lies on the circle

$$x^2 + y^2 = \frac{a^2b^2}{a^2 + b^2}.$$

7. Find the locus of a point, such that if normals be drawn from it to an ellipse, the sum of the squares of the tangents of the inclinations of the normals to the axis of x is constant.

See x. 50.

8. The circle $x^2+y^2=r^2$ is touched by a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; find the locus of the middle point of the chord.

9. Two similar and similarly situated conics have a common focus and axis. Prove that their common chord bisects the distance between their directrices.

10. If the tangents at points whose eccentric angles are α and β intersect on the diameter through γ, prove that $2\gamma-\alpha-\beta$ is an even multiple of two right angles; if on the diameter conjugate to the one through the point γ, shew that $2\gamma-\alpha-\beta$ is an odd multiple of two right angles.

11. If the normals at points α and β intersect on the diameter through the point γ, prove that

$$a^2 \tan\alpha \tan\beta \tan\frac{\alpha+\beta}{2}+b^2\tan\gamma=0;$$

if on the diameter conjugate to the one through the point γ, shew that

$$\tan\alpha \,.\, \tan\beta \,.\, \tan\gamma \,.\, \tan\frac{\alpha+\beta}{2}=\frac{b^2}{a^2}.$$

12. P, Q, R are points on an ellipse centre C. The diameter bisecting QR meets PQ, PR, and the curve in M, N and D respectively. Shew that

$$CM \,.\, CN=CD^2.$$

13. From a given point as centre circles are drawn cutting an ellipse. Shew that the locus of the middle points of the chords common to both curves passes through the feet of the normals drawn from the given point to the ellipse.

14. Find the locus of the poles of chords which subtend a right angle at the point (h, k).

15. CP, CD are two conjugate diameters of an ellipse; on PQ the tangent at P a length PQ is taken such that $PQ = mCD$. Find the locus of Q.

16. If P lies on

$$x^2 + y^2 = \sqrt{a^2x^2 + b^2y^2},$$

and Q on $$x^2 + y^2 = \sqrt{(c^2 - a^2)\,y^2 + (c^2 - b^2)\,x^2},$$

and if PQ subtends a right angle at the origin, then PQ is of constant length c.

17. If O be the origin, and if from any point P on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ a tangent be drawn to the circle $x^2 + y^2 = c^2$, the locus of the point of intersection of this tangent with the perpendicular to OP at O will be the conic

$$x^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) + y^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 1.$$

18. Two tangents intersect at right angles at P. If x be the abscissa of P, shew that the distances of P from the foci are

$$\sqrt{2a\,(a \pm ex)}.$$

19. Tangents are drawn to the ellipse at P and at Q, where the normal at P meets the curve again. If (h, k) be the co-ordinates of P, find the point where the tangents intersect.

20. Find the locus of the point of intersection of the tangents in the last example.

21. Find in the ellipse the locus of the foot of the perpendicular let fall from any point on its polar, the locus of the point being a diameter.

22. S and S' are the foci of an ellipse of eccentricity e. If R be a point on a chord through S' which divides it into segments subtending equal angles at S, shew that the point R will lie on the ellipse

$$x^2 + \frac{1+e^2}{1-e^2}\,y^2 = c^2,$$

where c is half the distance between the foci.

23. A circle touches an ellipse at a fixed point P; shew that the tangents at the two points where the circle meets the ellipse again intersect on a fixed diameter.

24. With a given point (α, β) as centre circles are described cutting the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; at the points of intersection tangents are drawn to the ellipse; shew that they will intersect on the curve

$$(a^2 - b^2)\,xy = (a^2\alpha y - b^2\beta x)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right).$$

Explain this result when $\beta = 0$.

25. A circle is inscribed in a quadrant of the ellipse so as to touch the curve and the axes. If (x_1, y_1) be the point of contact, p the perpendicular on the common tangent from the centre of the ellipse, shew that

$$\frac{1}{x_1} \sim \frac{1}{y_1} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \cdot p.$$

26. The pole of the chord of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, when the sum of the ordinates at its extremities $=b$, lies on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2y}{b};$$

when the sum of the abscissæ $=a$, it lies on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{a}.$$

27. Find the *foci* and *directrices* of

$$x^2+y^2=e^2(x\cos\alpha+y\sin\alpha-p)^2.$$

XIII. THE PARABOLA REFERRED TO THE VERTEX.

The co-ordinates of the point $(am^2, 2am)$ satisfy the equation of the parabola $y^2=4ax$; any point on the parabola may therefore be thus denoted; m is found to be the cotangent of the inclination of the tangent at the point to the axis of x. The advantage of using this notation is precisely the same as that attending the use of the eccentric angle in the ellipse; in any elimination it diminishes the number of equations by one and enables us to bring in trigonometrical methods which materially shorten the work.

The equation of the tangent at any point $(am^2, 2am)$ becomes $my=x+am^2$, which explains the value of m given above, when put in the form

$$y=\frac{1}{m}x+am.$$

1. Find the equation of the normal at the point

$$(am^2, 2am).$$

2. The equation of the chord of a parabola in terms of its middle point (α, β) is

$$y - \beta = \frac{2a}{\beta}(x - \alpha).$$

3. Find the locus of the middle points

(1) of all chords through the vertex,

(2) of all focal chords.

4. Find the locus of the middle points of chords which subtend a right angle at the vertex.

5. Find the co-ordinates of the intersection of two tangents to a parabola, at points whose ordinates are y_1 and y_2.

6. Find the area of the triangle contained by three points on a parabola whose co-ordinates are given.

7. Find the area contained by three tangents to a parabola; and shew that this area is one half of the triangle formed by their points of contact.

8. The area of the triangle formed by two tangents from a point (x_1, y_1) and its polar is

$$\frac{\{y_1^2 - 4ax_1\}^{\frac{3}{2}}}{2a}.$$

9. P and Q subtend an angle of 45°, firstly at the vertex, secondly at the focus. Find the locus of the intersection of the tangents at P and Q in each case.

10. Two tangents are drawn from P to the parabola $y^2=4ax$. If p_1, p_2 be the perpendiculars on them from the focus S, shew that $p_1p_2=a\,.\,SP$.

11. Three tangents are drawn to a parabola, intersecting in P, Q and R. If p_1, p_2, p_3 be the perpendiculars drawn from the focus S on QR, RP, PQ respectively, shew that $p_1\,.\,SP=p_2\,.\,SQ=p_3\,.\,SR$; hence prove that a circle passing through the intersections of three tangents to a parabola, passes through the focus.

12. Find the equation of the common tangents of

$$3x^2+2y^2=6$$

and

$$y^2=4\sqrt{2}\,.\,x.$$

13. Find the locus of the intersection of tangents with the perpendicular let fall on each from the focus.

14. The part of the polar of a point (x, y) intercepted by $y^2=4ax$ subtends a right angle at the point (h, k). Find the locus of (x, y). What does it become when $h=-a$?

15. The locus of the point, the perpendicular from the focus on the polar of which is constant, is a central conic, having its centre on the axis of the parabola, where the directrix of the same meets it.

16. Tangents are drawn from O to the parabola, S being the focus. On OS as diameter a circle is described cutting the tangents from O in P and Q. Shew that PQ is parallel to the directrix.

17. Through the vertex of a parabola a perpendicular is drawn to any tangent meeting it at P and the curve again in Q. Shew that $AP\,.\,AQ$ is constant.

18. Two tangents to a given parabola intersect in the directrix. From the vertex and focus respectively perpendiculars p_1, p_2, p_3, p_4 are drawn to these; shew that

$$p_1p_2p_3p_4 = a^4.$$

19. The equation of the chord of a parabola, joining the two points where the tangent makes angles α and β respectively with the axis of x, may be written

$$y = \frac{2x}{\cot\alpha + \cot\beta} + \frac{2a}{\tan\alpha + \tan\beta}.$$

20. If two tangents be drawn to a parabola making complementary angles with the axis, their chord of contact must pass through the foot of the directrix.

21. The intercept of the parabola $y^2 = 4ax$ on the polar of the point (x_1, y_1) is

$$\frac{\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}}{a}.$$

22. If p be the perpendicular from the vertex on a focal chord of length l; shew that

$$pl = 2a\sqrt{k^2 + 4a^2},$$

where k is the ordinate of the intersection of the tangents at the extremities of the chord.

23. The parabolas $y^2 = ax$ and $x^2 = 8ay$ cut at an angle $\tan^{-1}\frac{3}{5}$, and at right angles at the origin.

24. A circle is drawn with the focus of a parabola as centre and touches the directrix. Find the locus of the

point whose polar with respect to the parabola touches the circle.

25. The same circle being drawn as in (24), find the locus of the pole, with respect to the circle, of a line touching the parabola.

26. The parabolas $y^2 = 4ax$, $x^2 = 4by$ cut at an angle whose tangent is

$$\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}.$$

27. S is the focus, A the vertex of a parabola which is touched at P and Q by a circle through S; shew that

$$SP = SQ = 4SA.$$

28. Through the vertex of a parabola two lines are drawn, meeting the curve again in P and Q, and making angles α and β with the axis. Shew that the axis divides PQ in the ratio $\cot\alpha : \cot\beta$.

29. From a point O on the directrix tangents are drawn to a parabola touching it at P and Q. The bisector of the angle POQ meets PQ in R; shew that

$$\frac{\sqrt{2}}{OR} = \frac{1}{OP} + \frac{1}{OQ}.$$

30. Half the latus rectum is an arithmetic mean between the abscissae at the extremities of the chord of a parabola. Shew that the pole of the chord traces out the parabola

$$y^2 = 2a(x + 2a).$$

31. Tangents are drawn on the same side of the axis from points situated in the same, at distances $a, 3a, (7 + 4\sqrt{3})a$

to the left of the vertex. The sum of the inclinations to the axis of any two of these tangents is complementary to that of the third.

32. The triangle which has any chord of a parabola for base and its vertex at the vertex of the parabola has an area $x\sqrt{y^2 - 4ax}$, (x, y) being the pole of the chord.

33. The triangle which has any chord for base and its vertex at the focus has an area $(x + a)\sqrt{y^2 - 4ax}$, (x, y) being the pole of the chord.

Hence shew that the pole of a focal chord lies on the directrix, which therefore is the locus of the poles of chords through the focus.

34. If Δ_1, Δ_2, Δ_3 be the areas of triangles having a chord of a parabola as base, and the focus, the vertex and the pole of the chord respectively as vertices; shew that

$$(\Delta_1 - \Delta_2)^3 = 2a^4\Delta_3.$$

35. The tangents from a point O, whose co-ordinates are (h, k), meet the tangent at the vertex in P and Q; shew that the lengths OP, OQ are given by

$$\left\{h^2 + \left(\frac{k \pm \sqrt{k^2 - 4ah}}{2}\right)^2\right\}^{\frac{1}{2}}.$$

36. Two tangents touch a parabola at P and Q and intersect in O; find the locus of O if

(1) $SP + SQ$ be constant,

(2) $SP.SQ$ be constant,

S being the focus.

37. Find the focus and the directrix in each of the following parabolas:

(1) $(x-y)^2+2x-2y+1=0$,

(2) $(x-2y)^2-4x+38y+49=0$,

(3) $(3x-2y)^2=94x+128y-309$.

38. Ox, Oy are two lines at right angles. A is a fixed point on Ox, AP a line parallel to Oy. Through P, any point on AP, draw PQ parallel to Ox, and on it take Q so that POQ is a right angle. Shew that the locus of Q is a parabola, and that if PQ meet Oy in N, and T be the middle point of ON, then QT is the tangent at Q.

Normal Properties.

39. Normals at the extremities of a focal chord intersect at right angles.

40. The locus of the intersection of normals at the extremities of the focal chords of the parabola $y^2=4ax$ is the parabola $y^2=a(x-3a)$.

41. The area contained by a focal chord making an angle α with the axis and the two normals drawn at its extremities, is

$$4a^2 \operatorname{cosec}^3 \alpha.$$

42. Find the co-ordinates of the point where the normal at the point whose ordinate is y_1 meets the curve again.

43. If the normal at P meet the tangent at the vertex in Q and the curve again in R, shew that the axis

cuts QR into segments having the ratio of the abscissa of P to the semi-latus rectum.

44. If the normal at P meets the curve again in Q and the axis in R, shew that the abscissa of R is a mean proportional between those of P and Q; and if h and k are the ordinates of P and Q that

$$hk + 4a \,.\, AR = 0.$$

45. If the sum of the inclinations to the axis of normals from P to the parabola is ω, shew that P will lie on the straight line

$$y \cos \omega + (x - a) \sin \omega = 0.$$

46. Find the co-ordinates of the intersection of the normals at P and Q, in terms of the co-ordinates of the intersection of the tangents at P and Q.

47. Find the locus of intersection of tangents, when the normals at their points of contact intersect on the tangent at the vertex.

48. The normal at P meets the parabola again in Q; shew that the locus of intersection of the tangents at P and Q is the curve

$$(x + 2a) y^2 + 4a^3 = 0.$$

49. A chord AB is drawn perpendicular to the axis. If the tangent at A meets the normal at B in P, shew that P satisfies the equation

$$y^2 (a - x)^2 + \{y^2 - 2a (x - a)\} \{y^2 - 2x (x - a)\} = 0.$$

Why is this equation satisfied by the co-ordinates of the vertex and of the focus of the parabola?

50. The axis of the parabola, the normal at the point (x_1, y_1), and the straight lines

$$\frac{y}{x} = \frac{y_1}{x_1},$$

$$\frac{x}{x_1} - \frac{2ay}{y_1(2a + x_1)} + 1 = 0,$$

form a quadrilateral which is inscribable in a circle.

51. The locus of the point from which three normals can be drawn to a parabola, the product of the cosines of the inclinations to the axis being constant, is a circle having the focus as centre.

52. If normals from P meet a parabola in M, N, Q, the algebraical sum of the ordinates at these points is zero.

53. If normals be drawn from a given point to a parabola, the circle which passes through the points where they meet the curve passes also through the vertex.

54. OA, OB are two tangents to a parabola which intersect on the directrix. Through B draw a normal BC parallel to OA meeting the curve again in C; shew that $BC = 4OA$.

55. The equation of the parabola $y^2 = 4ax$, referred to the tangent and normal of the point as axes, at which the tangent makes an angle $\cot^{-1} m$ with the axis, becomes

$$(x + my)^2 + 8a(1 + m^2)^{\frac{3}{2}} y = 0.$$

56. The equation of the circle described on a focal chord as diameter is

$$x^2 + y^2 - ax\left(m^2 + \frac{1}{m^2}\right) - 2ay\left(m - \frac{1}{m}\right) - 3a^2 = 0,$$

where $2am$ is the ordinate of one extremity.

57. A circle is described on a focal chord as diameter; if y_1 be the ordinate of its centre, the ordinates of the remaining points of intersection of the circle with the parabola are given by

$$y^2 + 2ayy_1 + 12a^2 = 0.$$

58. Find the straight lines which are cut at right angles by each of the curves

$$y^2 = 4ax,$$

$$x^2 + 2y^2 = 54a^2.$$

59. If A, B, C be points on a parabola the normals at which meet at a point P; shew that

(1) If $SA + SB + SC = -3a$, the point P lies on the directrix.

(2) If $SA \cdot SB \cdot SC = k^3$, the point P lies on a circle having the focus as centre.

(3) If SB is the arithmetic mean between SA and SC, the locus of P will be the curve

$$\frac{ay^2}{2} + \left(\frac{2a - x}{3}\right)^3 = 0.$$

XIV. THE PARABOLA REFERRED TO TWO TANGENTS AS AXES.

[*The tangents are assumed to be at right angles unless the contrary is stated.*]

The equation

$$\frac{x^2}{a^2}-\frac{2xy}{ab}+\frac{y^2}{b^2}-\frac{2x}{a}-\frac{2y}{b}+1=0$$

represents a parabola since the highest terms are a perfect square. By putting $y=0$ we obtain two equal values $x=a$; by putting $x=0$ we have similarly $y=b$; therefore the axes touch the parabola at distances a and b from their point of intersection.

The curve represented is therefore a parabola referred to tangents of length a and b as axes.

If $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ be rationalized by squaring, the result will be found identical with the first equation, and in the following examples either form may be used as may be most convenient.

In $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ if we put $x=a\cos^4\theta$, $y=b\sin^4\theta$, we see that x and y satisfy the equation; any point on the curve may therefore be denoted by $(a\cos^4\theta, b\sin^4\theta)$.

When two tangents of a parabola intersect at right angles their intersection lies on the directrix, their chord of contact passes through the focus, and is perpendicular to the line joining the focus to the point of intersection. These properties will be found useful in what follows. For example: To find the co-ordinates of the focus.

The focus is the intersection of the chord of contact $\frac{x}{a}+\frac{y}{b}=1$ with the perpendicular to it through the origin $\frac{x}{b}-\frac{y}{a}=0$. Solving, we find the focus to be

$$\left(\frac{ab^2}{a^2+b^2},\ \frac{a^2b}{a^2+b^2}\right).$$

1. Find the equation of the axis of the parabola referred to two tangents of lengths a and b as axes.

2. Find the equation of the directrix, and the co-ordinates of the foot of the directrix.

3. Find the co-ordinates of the vertex.

4. Find the parameter of the parabola.

5. If h and k be the lengths of two tangents to a parabola, at right angles to one another, and $4a$ the parameter, shew that

$$\frac{1}{h^2}+\frac{1}{k^2}=\frac{1}{\{ahk\}^{\frac{2}{3}}}.$$

6. A parabola touches a given line AB at B, and A is a given point on the directrix. Find the locus of the focus.

7. Tangents OP, OP' from a point O in the directrix, meet the axis of the parabola in Q and Q'; shew that $QQ'.PP'=OP^2 \sim OP'^2$.

Explain the case where $OP=OP'$.

8. Tangents OP, OP' are drawn from a point in the directrix, the normals at P and P' meet the axis in Q and Q'; shew that

$$\{PQ^2+P'Q'^2\}^{\frac{1}{2}}=\frac{2OP.OP'}{PP'}.$$

9. From a point O on the directrix, a tangent OP is drawn to a parabola; the normal at P meets the axis in Q and the curve again in R; shew that

$$\frac{1}{OP^2}+\frac{16}{PR^2}=\frac{8}{PR.PQ}.$$

10. The condition that the line

$$\frac{x}{h}+\frac{y}{k}=1$$

should touch the parabola

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1,$$

is $$\frac{h}{a}+\frac{k}{b}=1.$$

11. The circle circumscribing the triangle formed by two tangents at right angles and $\frac{x}{h}+\frac{y}{k}=1$ will pass through the focus if this last line touches the parabola.

12. A parabola touches two fixed straight lines AB, AC at right angles at B and C; $AB+AC$ being constant, find the locus of the focus.

13. Parabolas are described touching two straight lines at right angles to one another. If the chords joining the points of contact in each case are parallel, the locus of the vertices of the parabolas is a straight line.

14. Parabolas touch two fixed straight lines at right angles to one another. The chord joining the points of contact in each case always passes through the fixed point (h, k). Shew that the locus of the focus is a circle, passing through (h, k).

15. Find in (14) the locus of the vertex.

16. In the parabola

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1,$$

referred to two tangents inclined to one another at an angle ω, the co-ordinates of the focus are given by

$$ax = by = \frac{a^2b^2}{a^2 + b^2 + 2ab \cos \omega}.$$

17. The directrix is given by

$$x(a + b \cos \omega) + y(b + a \cos \omega) = ab \cos \omega.$$

18. Given the point of contact of one of two fixed tangents to a parabola, shew that the focus describes a circle passing through the intersection of the tangents, and touching one of them.

19. Establish (11) when the tangents are inclined at an angle ω.

20. If two tangents OP, OP' meet the directrix in Q and Q' respectively, shew that

$$\frac{1}{OQ} + \frac{1}{OQ'} = \left(\frac{1}{OP} + \frac{1}{OP'}\right)(\sec \omega + 1).$$

21. If QQ' be the intercept on the directrix of tangents of lengths a and b respectively and inclined at an angle ω, shew that

$$QQ' = \frac{ab \,.\, \sin \omega \cos \omega}{(a + b \cos \omega)(b + a \cos \omega)} \,.\, \{a^2 + b^2 + 2ab \cos \omega\}^{\frac{1}{2}}.$$

XV. THE PARABOLA REFERRED TO POLAR CO-ORDINATES, THE VERTEX BEING THE POLE.

The polar equation of the parabola referred to the vertex as pole and the axis as initial line is

$$r \sin^2 \theta = 4a \cos \theta.$$

This follows from substitution in the equation $y^2 = 4ax$.

1. The equation of the chord joining two points θ_1, θ_2 on the parabola

$$r = \frac{4a \cos \theta}{\sin^2 \theta}$$

is $$\frac{4a}{r} = (\tan \theta_1 + \tan \theta_2) \sin \theta - \tan \theta_1 \tan \theta_2 \cos \theta.$$

Deduce the equation of the tangent at the point θ_1; viz.

$$\frac{4a}{r} = 2 \tan \theta_1 \sin \theta - \tan^2 \theta_1 \cos \theta.$$

2. The tangent at any point intersects the perpendicular upon it from the vertex in the curve

$$r \cos \theta + a \sin^2 \theta = 0.$$

3. The locus of the intersection of the tangent at any point, with the perpendicular to the radius vector from the vertex to the point of contact is the curve

$$r \cos \theta (1 + \sin^2 \theta) + 4a \sin^2 \theta = 0.$$

4. Two radii vectores AP, AQ are drawn at right angles through the vertex; R is the middle point of PQ. If θ_1 and θ_2 be the angles which AP and AR respectively make with the axis of the parabola, shew that

$$\tan^3 \theta_1 = \tan (\theta_1 - \theta_2).$$

5. Two radii vectores r_1, r_2 are drawn at right angles through the vertex, r is the length of the chord joining their extremities. Shew that

$$r = \frac{r_1 r_2}{p^3} \sqrt{r_1^2 r_2^2 - 3p^4},$$

where p is the parameter of the parabola.

XVI. THE PARABOLA REFERRED TO THE FOCUS AS POLE AND THE AXIS AS INITIAL LINE.

The equation to the parabola in this form is

$$\frac{2a}{r} = 1 + \cos\theta;$$

of the tangent at the point α,

$$\frac{2a}{r} = \cos\theta + \cos(\theta - \alpha),$$

which can be obtained from the equation corresponding to it in the ellipse, by putting $e = 1$.

1. Tangents at the extremities of a focal chord PP' of a parabola meet the axis in Q and Q'; shew that

$$SP \sim SP' = QQ'.$$

2. Find the length of the radius vector of the point of intersection of tangents whose points of contact are

$$\alpha + \beta,\ \alpha - \beta.$$

3. The tangents at the points $\alpha + \beta$, $\alpha - \beta$ being drawn, find the polar equation of the circle described on the line joining their intersection to the focus as diameter.

4. If the circle pass through an extremity of the latus rectum the intercept which it makes on the axis will be $2a \cot \alpha$.

5. Find the locus of the intersection of tangents to a parabola

(1) when the sum,

(2) when the product of the focal distances of the points of contact is given.

6. The equation of the normal at α to the parabola is

$$\frac{a}{r} = \cot\frac{\alpha}{2} \cdot \cos\frac{\alpha}{2} \cdot \sin\left(\theta - \frac{\alpha}{2}\right).$$

7. Shew that the angular co-ordinate of the point of intersection of normals at the extremities of a focal chord making an angle α with the axis is given by the equation

$$\tan\left(\theta - \frac{\alpha}{2}\right) + \tan^3\frac{\alpha}{2} = 0.$$

8. If tangents be drawn from a point on the latus rectum and P, P' be the points of contact; shew that half the latus rectum is a harmonic mean between SP and SP'.

9. Tangents are drawn from a point on the latus rectum, their chord of contact subtending an angle 2α at the focus; find their lengths.

10. If l and l' be the lengths found in the last example, shew that

$$\frac{l^4 l'^4}{a^2} = (l^2 - l'^2)^2 (l^2 + l'^2).$$

11. Two parabolas having the same focus, and axes inclined at an angle α, intersect. How many real points of intersection are there? Trace the curves for various values of α between 0 and π.

12. The angular co-ordinates β and γ of the points of contact of the common tangents to two confocal parabolas of latera recta $4a$, $4b$, with the axis of the second inclined at an angle α to that of the first, are given by

$$\frac{a}{b}=\frac{1+\cos\beta}{\cos\alpha+\cos\gamma}=\frac{\sin\beta}{\sin\gamma-\sin\alpha}.$$

13. Shew that the common tangent to the parabolas in (12) is represented by the equation

$$\frac{a^2-2ab\cos\alpha+b^2}{r}=\{a\sin(\theta+\alpha)-b\sin\theta\}\sin\alpha.$$

Explain the result when $\alpha=0$.

14. Two confocal parabolas have their axes in opposite directions; shew that the tangents to each curve at the points of intersection make complementary angles with the direction of the axis.

15. Find the locus of the point of intersection of the normal at any point with the circle described on the radius vector to the point as diameter, the focus being the pole.

16. The point of intersection of the perpendicular to SP at S with the normal at P lies on the locus

$$r=a\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\sec^2\left(\frac{\theta}{2}-\frac{\pi}{4}\right).$$

17. PT, a tangent at P, meets the axis in T. Find the locus of the centre of the circle inscribed in the triangle SPT.

18. If the bisector of the angle PTS in (17) meet SP in M, find the locus of M.

19. If from any point on the conic

$$r\left(\frac{2a}{c} - \sin^2\theta\right) = 2a\cos\theta$$

tangents be drawn to the parabola

$$\frac{2a}{r} = 1 + \cos\theta,$$

touching at P and P', shew that c is a harmonic mean between SP and SP'.

20. Shew that pairs of tangents to a parabola whose points of contact subtend a constant angle at the focus intersect on a confocal conic.

21. AB is a stretched, inelastic string passing through a ring at S; B moves along a fixed line through S, A always lies in the perpendicular drawn to the fixed line through B. Find the locus of A.

22. Tangents at the points $\alpha+\beta$, $\alpha-\beta$ to the parabola $\frac{2a}{r} = 1 + \cos\theta$ intersect on a confocal coaxal parabola; shew that its latus rectum is

$$4a \cdot \frac{1+\cos\alpha}{\cos\alpha+\cos\beta}.$$

23. Tangents from a point to a parabola are of lengths l_1 and l_2; if their chord of contact subtends a constant angle 2α at the focus, shew that

$$l_1^2 + l_2^2 - 2l_1l_2\cos\alpha = \frac{l_1^{\frac{4}{3}}l_2^{\frac{4}{3}}}{a^{\frac{2}{3}}} . \sin^{\frac{4}{3}}\alpha.$$

XVII. THE HYPERBOLA REFERRED TO THE CENTRE AS ORIGIN.

1. The equation of the tangent reduces to the form

$$x\cos\alpha+y\sin\alpha=p,\text{ if }p=\sqrt{a^2\cos^2\alpha-b^2\sin^2\alpha}.$$

2. The locus of the intersection of tangents which cut at right angles is the circle

$$x^2+y^2=a^2-b^2.$$

Note. The eccentric notation for the hyperbola is

$$x=a\sec\theta,\quad y=b\tan\theta.$$

3. The equation of the chord joining two points θ and ϕ on the hyperbola is

$$\frac{x}{a}\cos\frac{\theta-\phi}{2}-\frac{y}{b}.\sin\frac{\theta+\phi}{2}=\cos\frac{\theta+\phi}{2},$$

the tangent at the point θ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

4. Find the co-ordinates of the point of intersection of the tangents at points α and β on the hyperbola.

5. Find the equation of a chord of the hyperbola in terms of its middle point (x_1, y_1).

6. Prove that the equations of the common tangents to the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

$$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,$$

are $$x\pm y=\sqrt{a^2-b^2}.$$

7. Find the locus of the point of intersection of the tangent at any point with the perpendicular on it

(1) from the centre,

(2) from a focus.

8. The product of the perpendiculars from two fixed points on a hyperbola on any tangent is in a constant ratio to the square of the perpendicular on it from the point of intersection of the tangents at those points.

9. In an equilateral hyperbola of transverse semi-axis AC and centre C, the angle between CP and the tangent at any point P is

$$\sin^{-1}\left(\frac{CA^2}{CP^2}\right).$$

10. A straight line touches a circle passing through the foci and having its centre at the centre of the hyperbola. Find the locus of its pole with respect to the latter curve.

11. From the centre C of an equilateral hyperbola a perpendicular CP is drawn on the tangent at any point. The perpendicular meets the curve again at P'.

Shew that $CP\,.\,CP'=a^2$, where a is half the transverse axis.

12. In the case of the ordinary hyperbola with semi-axes a and b, shew that the relation in (11) is

$$CP^2 = a^2 - b^2 + \frac{a^2b^2}{CP'^2}.$$

13. Find the common tangents to the curves

$$y^2 = 4ax,$$
$$x^2 - 12y^2 = 24a^2.$$

14. The line

$$y = mx + \frac{p}{m}$$

will touch the parabola $y^2 = 4px$, and the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

provided that

$$m^2(a^2m^2 - b^2) = p^2.$$

15. A point P on an equilateral hyperbola being joined to the foci S and S', the centre being C, shew that the difference of the angles PSS' and $PS'S$ is double the angle between CP and SS'.

16. The chord of an equilateral hyperbola touches the auxiliary circle at Q. If the co-ordinates of P, the middle point of the chord, be (x_1, y_1), shew that

$$\sin CPQ = \frac{a^2}{x_1^2 - y_1^2}.$$

17. In a hyperbola if a circle be described with a focus as centre, and half the minor axis as radius, shew that the polar of any point on the hyperbola with respect to this circle will touch the auxiliary circle.

18. Shew that the equation of the normal assumes the form $x\cos\alpha + y\sin\alpha = p$ if

$$p = \frac{(a^2+b^2)\sin\alpha\cos\alpha}{\sqrt{a^2\sin^2\alpha - b^2\cos^2\alpha}}.$$

Explain the value of p when

$$\alpha = \tan^{-1}\frac{b}{a}.$$

19. The perpendicular from the centre on a normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets it in P. If P lies on the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, shew that

$$\frac{PN}{CN} + \frac{CN}{PN} = \frac{a}{b} + \frac{b}{a},$$

where PN is the ordinate of P.

20. Find the equation of the normal in terms of the eccentric angle θ.

21. Shew that the sum of the eccentric angles at the points where normals from a given point meet the hyperbola is an odd multiple of two right angles.

22. The perpendicular from the centre on any normal to an equilateral hyperbola does not meet the curve in real points.

23. Two straight lines intersect at O; from a fixed point A in one, a perpendicular AC is drawn on the other; DB is a perpendicular from any point D in CO on OA. Shew that the intersection of AD and BC is a rectangular hyperbola.

24. If the focus of an hyperbola be the centre of a circle with radius equal to the semi-axis minor, the circle

will touch the asymptotes where the directrix meets them.

25. Two points A and B are fixed. From A a perpendicular AP is drawn on the polar of B with respect to an equilateral hyperbola, centre C; from B a similar perpendicular BQ is drawn on the polar of A. Shew that

$$AP . BC = BQ . AC.$$

26. From a given point normals are drawn to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Shew that the tangents of their inclinations to the axis of x are given by

$$m^4b^2x^2 - m^3 . 2b^2xy + m^2 \{b^2y^2 - a^2x^2 + (a^2+b^2)^2\} \\ + m . 2a^2xy - a^2y^2 = 0.$$

27. Find the locus of a point, such that the sum of the squares of the tangents of the inclinations to the axis of x of the four normals from that point to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is constant.

28. PQ is a chord of an equilateral hyperbola normal at P, C is the centre; shew that

$$PQ^2 = CQ^2 + 3CP^2.$$

29. PNP' is a double ordinate to an ellipse whose vertices are A and A'. AP and $A'P'$ intersect in Q. Find the locus of Q.

30. The tangent to

$$y(y-x) = x(x-a)$$

at one of the points of intersection of the conic with the axis of x, is parallel to one of the chords of intersection with the parabola $y^2 + ax = 0$.

XVIII. THE HYPERBOLA REFERRED TO ITS ASYMPTOTES AS AXES.

The equation of the hyperbola in this form is

$$xy = k^2 = \frac{a^2 + b^2}{4},$$

where a and b are the semi-axes. When the axes are rectangular, the hyperbola is rectangular, $a = b$, and the equation becomes $xy = \frac{a^2}{2}$.

1. A straight line moves so as to cut off a constant area from two intersecting lines; shew that the middle point of this line will trace out a hyperbola of which the two intersecting lines are the asymptotes.

2. Find the co-ordinates of the intersection of

(1) the tangents,

(2) the normals,

at (x_1, y_1) (x_2, y_2) to $xy = k^2$.

3. A parabola has one of the asymptotes of a rectangular hyperbola as axis, the other as tangent at the vertex. Find the locus of the poles, with respect to the hyperbola, of tangents to the parabola, the latus rectum of the latter being equal to $2\sqrt{2}$ times the transverse axis of the former.

4. The equation of the chord in terms of its middle point (α, β) may be written

$$\alpha y + \beta x = 2\alpha\beta.$$

5. A tangent to the parabola

$$(x+y)^2 = 4ax$$

cuts the hyperbola $xy = c^2$ in two points P and Q. The locus of the middle point of PQ is the hyperbola

$$2y(x-y) = ax.$$

6. Chords of $xy = k^2$ are drawn through (α, β) and bisected there. Shew that the point of intersection of the tangents at P and Q lies on another hyperbola having the same asymptotes.

7. If two fixed points P, Q on a hyperbola be joined to any variable point R on the same, the portions which the lines PR, QR intercept on each asymptote are constant.

8. If normals be drawn to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, from a point

$$\left(\frac{a^2-b^2}{a^2} h,\ \frac{a^2-b^2}{b^2} k\right),$$

these normals will meet the curve where an equilateral hyperbola meets it, whose centre is at $(-h, k)$ and whose asymptotes are parallel to the axes.

9. Any fixed point in the line that cuts off a constant area from two other lines traces out a hyperbola, of which these lines are asymptotes.

10. The asymptotes of a hyperbola touch a parabola at A and B respectively. The pole, with respect to the hyperbola, of any tangent to the parabola will lie on a straight line through the middle points of CA and CB, where C is the point of intersection of the asymptotes.

11. Two straight lines OP, OQ intersect in O. PQ is any straight line passing through a fixed point R. If on PQ or on PQ produced $P'Q$ is taken equal to $m.PQ$, shew that the locus of P' is a hyperbola, with its asymptotes parallel to the two fixed lines.

12. A circle touches two straight lines intersecting at C. A variable tangent to the circle meets the lines in A and B. If the perimeter of ABC be constant, shew that any point, dividing AB in a given ratio, traces out a hyperbola, of which CA and CB are the asymptotes.

13. Shew that the co-ordinates of the foci of

$$xy = k^2 = \frac{a^2 + b^2}{4}$$

are found from the equations

$$x = y = \pm \frac{a^2 + b^2}{2a}.$$

XIX. THE HYPERBOLA REFERRED TO POLAR CO-ORDINATES, THE FOCUS BEING THE POLE.

The equation of the curve is $\frac{l}{r} = 1 + e\cos\theta$, where e is greater than unity. The equations of the tangent and normal can be found in precisely the same way as for the ellipse.

To find the asymptotes of $\frac{l}{r} = 1 + e\cos\theta$.

The asymptotes are straight lines which touch the curve at infinity; making r infinite we obtain from

$$\cos\theta_1 = -\frac{1}{e}, \text{ or } \sin\theta_1 = \pm\sqrt{\frac{(e^2-1)}{e^2}},$$

the angular co-ordinates of the points of contact.

Substituting in $\frac{l}{r} = e\cos\theta + \cos(\theta - \theta_1)$ the values of θ_1 just found, the equations of the asymptotes are obtained in the form

$$\frac{e}{r} = \frac{\cos\theta}{a} \pm \frac{\sin\theta}{b}.$$

1. The angle between the asymptotes is

$$\cos^{-1}\frac{2-e^2}{e^2}.$$

2. A focal radius vector meets the asymptote

$$\frac{e}{r} = \frac{\cos\theta}{a} + \frac{\sin\theta}{b}$$

in P and the straight line

$$\frac{x}{a} - \frac{3y}{b} = e$$

in Q. On PQ, R is taken so that SR is a harmonic mean between SP and SQ. Find the locus of R.

3. If the lines joining P, a point on an asymptote, to the foci S and S' be at right angles, shew that the angle

$$PSS' = \tfrac{1}{2}\tan^{-1}\left(\pm\frac{b}{a}\right);$$

S being the pole.

4. Any focal radius vector meets an asymptote in P; take Q so that

(1) $PQ=k$,
(2) $SP \, . \, SQ=k^2$ where k is a constant,

and find the locus of Q in each case.

5. If SP be drawn through the focus parallel to an asymptote, meeting the curve in P, shew that its length is one fourth of the latus rectum.

6. In the last example, if θ be the angle between SP and the normal at P, and ϕ the angle between the asymptotes, shew that

$$\tan\theta=\tfrac{1}{2}\tan\frac{\phi}{2}.$$

7. A parabola and a hyperbola have a focus in common and the other focus of the hyperbola lies on the axis of the parabola. Find the condition that an asymptote of the latter curve should touch the former; and find the co-ordinates of the point of contact.

8. Two points P and P' on an asymptote subtend a right angle at a focus S. Shew that

$$\frac{1}{SP^2}+\frac{1}{SP'^2}=\frac{1}{b^2}.$$

9. A radius vector, drawn from the focus to the point whose angular co-ordinate is α, subtends right angles at points on the asymptotes, whose angular co-ordinates are given by the equation in θ

$$\cos\alpha-\cos\beta=\sin\beta\cos(\alpha-\theta)\sin(\beta\mp\theta),$$

where β is the half angle between the asymptotes.

10. The perpendicular from a focus S on an asymptote meets the curve in P and P'; shew that

$$\frac{1}{SP \, . \, SP'} + \frac{1}{b^2} = \frac{1}{l^2};$$

where l is the semi-latus rectum.

11. ABC is a right-angled triangle on a constant hypothenuse AB. Find the locus of a point P in BC, at a vertical distance

$$\frac{AC \, . \, BC \, . \, AB + \mu AC \, . \, BC^2 - k \, . \, AC \, . \, AB}{AB \, (AB + \mu BC)}$$

below C, k and μ being constants.

12. A chord PQ of the hyperbola

$$\frac{l}{r} = 1 + \sqrt{2} \cos \theta$$

subtends a right angle at the focus S; shew that the centre of the circle inscribed in SPQ describes the curve

$$\frac{l}{2r} = \cos \frac{\theta}{2} \left(1 + \sqrt{2} \cos \frac{\theta}{2}\right).$$

XX. THE GENERAL EQUATION OF THE SECOND DEGREE.

The equation which will be used is the same as in *Salmon's Conic Sections:*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The axes are assumed rectangular unless the contrary is stated.

1. Find the conditions that the general equation of the second degree may represent

(1) a circle,
(2) a parabola,
(3) an equilateral hyperbola.

2. Shew that the centre of the conic given by an equation in the general form is found from the intersection of

$$\left.\begin{array}{r} ax + hy + g = 0 \\ hx + bx + f = 0 \end{array}\right\}.$$

Explain the values of the co-ordinates of the centre when

$$h^2 = ab.$$

Note. Let O be the origin and let the axes cut the conic in AA', BB' respectively, and be inclined at any angle.

Then in the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

putting $x = 0$, we obtain

$$BO \,.\, OB' = -\frac{c}{b};$$

putting $y = 0$, we have

$$AO \,.\, OA' = -\frac{c}{a};$$

$$\therefore \frac{AO \,.\, OA'}{BO \,.\, OB'} = \frac{a}{b},$$

but neither a nor b are altered by the axes being moved parallel to themselves; hence we obtain the theorem: that if through any point O straight lines are drawn meeting the conic in AA', BB' respectively, then will the ratio

$$\frac{AO \,.\, OA'}{BO \,.\, OB'}$$

be *independent of the position of O*, so long as the directions of the transversals remain the same.

It follows therefore, if CP and CD be semi-diameters parallel to the transversals, that

$$\frac{AO \,.\, OA'}{BO \,.\, OB'} = \frac{CP^2}{CD^2}.$$

3. If four points on a conic lie on a circle the lines joining them in pairs will be equally inclined to the axis of the conic.

4. All conics circumscribing a quadrilateral will have parallel axes, if the quadrilateral is inscribable in a circle.

5. Apply the preceding note to prove the following. If QQ', any chord of a parabola, be bisected by its diameter PV at V, then will $QV^2 = 4SP \,.\, PV$, S being the focus.

6. The eccentricity e of a conic, expressed by the general equation, is given by

$$\frac{e^4}{1-e^2} = \frac{(a+b)^2}{ab-h^2} - 4.$$

7. If the angle between the axes is ω, the eccentricity is given by

$$\frac{e^4}{1-e^2} = \frac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin^2\omega} - 4.$$

8. Find the locus of the point, the perpendicular from the origin on the polar of which is of constant length; the conic being represented by the general equation.

9. A tangent is drawn to a fixed circle, and its pole with respect to another fixed circle is taken; shew that the locus of this pole is a conic section.

10. Find the condition that the two circles

$$x^2 + y^2 + 2ax + 2by + c = 0,$$
$$x^2 + y^2 + 2bx - 2ay + d = 0,$$

should cut orthogonally.

11. Find the locus of the centre of a circle which cuts two given circles orthogonally.

12. Find the length of the intercept cut off on

$$x \cos \alpha + y \sin \alpha - p = 0$$

by $$x^2 + y^2 + 2gx + 2fy + c = 0.$$

13. Find the condition that the intercept on the chord $x \cos \alpha + y \sin \alpha = p$, of the circle $x^2 + y^2 = 2rx$ should subtend a right angle at the point $\{r(1+m), r(1-m^2)^{\frac{1}{2}}\}$; where m is a proper fraction.

14. Shew that the condition that the intercept on

$$x \cos \alpha + y \sin \alpha = p$$

made by the circle in (12) should subtend a right angle at the origin is

$$p^2 + p(g \cos \alpha + f \sin \alpha) + \frac{c}{2} = 0.$$

15. Find the locus of the middle point of the chord in the last example, and shew that this locus is identical with that described by the foot of the perpendicular let fall from the origin on the chord.

16. Find the condition that $lx + my = 1$ should touch the parabola

$$(ax - by)^2 - 2(a^2 + b^2)(ax + by) + (a^2 + b^2)^2 = 0,$$

and shew, that if this line meets the axes in P and Q, then PQ will subtend a right angle at the point (a, b) when the line touches.

17. The conic $ay^2 + bxy + x^2 = 7x$ meets the axis of x in A, and the lines $x \pm y = 0$ in P and Q respectively. Find a and b so that AP may be parallel to the axis of y, and AQ may make an angle of 45° with it.

18. Shew that the conditions that the conics

$$ax^2 + 2hxy + by^2 = 1,$$
$$a'x^2 + 2h'xy + b'y^2 = 1,$$

should cut at right angles are

$$\frac{aa' + hh'}{a - a'} = \frac{h(a' + b') + h'(a + b)}{2(h - h')} = \frac{bb' + hh'}{b - b'}.$$

Explain this result when the conics become concentric.

Note. If $l_1 = 0$, $l_2 = 0$, $l_3 = 0$, $l_4 = 0$ be the equations of four straight lines; $l_1 = 0$ standing for $y = m_1x + c_1$, &c.; the equation of a conic passing through their four points of intersection taken in order is of the form $l_1l_3 = kl_2l_4$; for this is of the second degree and is satisfied by any pair of values obtained by taking $l_1 = 0$, $l_2 = 0$, for instance: k being an undetermined constant, the conic can fulfil one more condition.

19. A straight line cuts off intercepts OA and OB on the axes, of lengths a and b respectively. On OA a length c is taken and a parallel CD drawn to AB through

its extremity. Find the equation of the conic circumscribing $ABCD$ and passing in addition through the point $(b-a+c, b)$.

20. Find the locus of the centres of conics circumscribing the quadrilateral $ABCD$, where OAB is a given triangle, and CD any parallel to the base.

21. Find the locus of the centres of conics touching the axes of x and y where $y = mx + c$ meets them.

22. Two lines cut off intercepts a and b, a' and b' respectively, from rectangular axes. The condition, that all the conics circumscribing the quadrilateral they form with the axes should be equilateral hyperbolas, is $aa' + bb' = 0$.

23. If the axes in the last example be oblique, find the value of the constant k (see note p. 92), which will make the circumscribing conic an equilateral hyperbola.

24. Find the locus of the centres of equilateral hyperbolas circumscribing the quadrilateral, formed by the lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$, with rectangular axes.

25. Find the locus of the centres of the equilateral hyperbolas that can be circumscribed about a given right-angled triangle.

26. Find the area of the conic

$$ax^2 + 2hxy + by^2 = c.$$

27. Find the area of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

28. If A be the area of a conic whose equation is expressed in its general form, Δ the area of the triangle cut

off from the two principal diameters by the polar of the origin, shew that

$$2\Delta = \frac{(ab - h^2)^2}{(hf - bg)(gh - af)} \cdot \left(\frac{A}{\pi}\right)^2.$$

29. The equation of the directrix of the parabola

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$$

is $$\beta x - \alpha y + \frac{c(\alpha^2 + \beta^2) - g^2 - f^2}{2(\beta g - \alpha f)} = 0;$$

and the co-ordinates of the foci are

$$\left(-\frac{g + \beta k}{\alpha^2 + \beta^2},\ \frac{-f + \alpha k}{\alpha^2 + \beta^2}\right),$$

where $$k = \frac{c(\alpha^2 + \beta^2) - g^2 - f^2}{2(\beta g - \alpha f)},$$

rectangular axes being used.

30. Shew that the equation of the tangent at the vertex of the parabola

$$\frac{x^2}{a^2} - \frac{2xy}{ab} + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0$$

referred to rectangular axes is

$$\frac{x}{b} + \frac{y}{a} = \frac{ab}{a^2 + b^2}.$$

31. Given a focus, a tangent, and the eccentricity of a conic, shew that the locus of the other focus is a circle.

32. Shew that the locus of the middle points of straight lines, of given length, terminated by two given straight lines, is an ellipse.

XXI. ABRIDGED NOTATION.

The general equation of the second degree in Trilinear co-ordinates is

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta = 0.$$

From this equation, by giving certain values, zero among the number, to the constants, we can deduce the following special cases which are important.

$$\text{(I)} \quad \alpha\gamma = k\beta\delta,$$

where δ is a straight line whose equation is of the form $l\alpha + m\beta + n\gamma = 0$.

$$\text{(II)} \quad \alpha\gamma = k\beta^2.$$

$$\text{(III)} \quad l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

$$\text{(IV)} \quad \sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0,$$

or, as it may be written in an expanded form,

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2lm\alpha\beta - 2mn\beta\gamma - 2ln\gamma\alpha = 0.$$

$$\text{(V)} \quad l^2\alpha^2 \pm m^2\beta^2 \pm n^2\gamma^2 = 0.$$

I. $\alpha\gamma = k\beta\delta$ is satisfied when $\alpha = 0$, $\beta = 0$; its locus therefore passes through the intersection of the lines $\alpha = 0$, $\beta = 0$. Similarly it can be shewn to pass through the remaining intersections of $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, taken in order; and it is of the second degree, it is therefore a conic, circumscribing the quadrilateral formed by these four lines.

II. In $\alpha\gamma = k\beta^2$ putting $\alpha = 0$, we obtain two equal values from $\beta^2 = 0$, which shews that $\alpha = 0$ touches the conic where $\beta = 0$ meets it; so $\gamma = 0$ touches the conic where $\beta=0$ meets it; this form therefore denotes the conic of which two sides of the triangle of reference are tangents, while the third is their chord of contact.

III. $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ is satisfied when $\alpha=0$, $\beta=0$; its locus therefore passes through the vertex (α, β) of the triangle of reference. Similarly it may be shewn to pass through the two other vertices; it is therefore a conic, circumscribing the triangle of reference.

IV. Writing the equation of the conic in the form

$$(l\alpha-m\beta)^2=n\gamma\,(2l\alpha+2m\beta-n\gamma),$$

we see that it reduces to form II.; therefore

$$\gamma=0,\ \ 2l\alpha+2m\beta=n\gamma$$

touch the conic where $l\alpha=m\beta$ meets it. The above equation therefore represents a conic inscribed in the triangle of reference, since $\alpha=0$ and $\beta=0$ can be proved to be tangents in a similar way.

Note. It is important to remember that $l\alpha-m\beta=0$ and two similar equations, are the lines joining the vertices, to the opposite points of contact.

V. Take the equation in the form $n^2\gamma^2=l^2\alpha^2-m^2\beta^2$; this shews that $\gamma=0$ is the polar of the point of intersection of the lines $l\alpha\pm m\beta=0$; but this point of intersection is the vertex opposite to $\gamma=0$. Similar reasoning holds for $\alpha=0$ and for $\beta=0$. The triangle of reference is therefore self-conjugate with respect to the conic, expressed in this form.

The following method of finding the chord and tangent can be applied to any of the cases.

To find the equation of the chord joining the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

$$l(\beta - \beta_1)(\gamma - \gamma_2) + m(\gamma - \gamma_1)(\alpha - \alpha_2) + n(\alpha - \alpha_1)(\beta - \beta_2)$$
$$= l\beta\gamma + m\gamma\alpha + n\alpha\beta$$

represents a straight line since it is of the first degree; but it is satisfied if for α, β, γ we put α_1, β_1, γ_1, or α_2, β_2, γ_2; it is therefore a straight line, joining the two points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

To obtain the tangent at $(\alpha_1, \beta_1, \gamma_1)$ put $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$, $\gamma_2 = \gamma_1$ in the equation of the chord; when we have

$$l(\beta\gamma_1 + \gamma\beta_1) + m(\gamma\alpha_1 + \alpha\gamma_1) + n(\alpha\beta_1 + \beta\alpha_1) = 0.$$

Note. It is easily seen that the equation of the tangent at the vertex A is $n\beta + m\gamma = 0$, and similar equations hold for the tangents at the other vertices.

1. To find the equation of the circle circumscribing the triangle of reference.

2. Find the equation of the circle inscribed in the triangle of reference.

3. Find the equation of the ellipse touching at the middle points of the sides of the triangle of reference.

4. Find the equation of the conics touching the sides of the triangle of reference at the points

(1) where the bisectors of the angles meet the opposite sides,

(2) where the perpendiculars from the vertices meet the opposite sides.

5. Find the equation of the circle, with respect to which the triangle of reference is self-conjugate.

6. Shew how to find the equation of the circle described on the line joining $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ as diameter; and apply the method to find that of the circle described on the side AB of the triangle of reference as diameter.

7. The tangents at the vertices to the circle circumscribing the triangle of reference meet the opposite sides in points which lie on the straight line

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0.$$

8. Shew that the equation of every circle may be written in the form

$$(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=(a\alpha+b\beta+c\gamma)\,(l\alpha+m\beta+n\gamma).$$

Hence shew how to find the equation of the radical axis of two circles.

9. Find the equation of the radical axis of the circles

$$\alpha\beta=\gamma\,(\gamma\cos C-\alpha\cos A-\beta\cos B),$$

$$\beta\gamma=\alpha\,(\alpha\cos A-\beta\cos B-\gamma\cos C).$$

10. The circle described on AB as diameter cuts CA in D, CB in F. Shew that the equation of the circle circumscribing the triangle CDF is

$$c\alpha\beta=(a\alpha+b\beta)\,(\alpha\cos A+\beta\cos B-\gamma\cos C).$$

Prove that this circle passes through the centre of perpendiculars of the triangle ABC.

11. If (α, β, γ) be the co-ordinates of the centre of the circumscribing circle, shew that

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = \tfrac{1}{4}abc.$$

12. Shew that the equation of the tangent at any point of the circumscribing circle is of the form

$$\left(\frac{\gamma}{c} + \frac{\alpha}{a}\right)\tan^2\theta - \frac{2\alpha}{a}\tan\theta + \left(\frac{\alpha}{a} + \frac{\beta}{b}\right) = 0,$$

where θ varies from point to point.

13. A variable tangent to the circle circumscribing the triangle of reference meets the tangents at the vertices B and C in $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ respectively; shew that

$$\frac{\beta_1\beta_2}{b^2} + \frac{\gamma_1\gamma_2}{c^2} + \frac{3\beta_2\gamma_1 - \gamma_2\beta_1}{bc} = 0.$$

If the variable tangent meets the tangent at A in $(\alpha_3, \beta_3, \gamma_3)$, shew that

$$\frac{\alpha_1\beta_2\gamma_3}{\alpha_2\beta_3\gamma_1} = -1.$$

Note. The two points on the line of centres where the common tangents to two circles intersect, are called the centres of similitude of the two circles.

14. With the vertices A, B, C of the triangle of reference as centres, circles of radii r_1, r_2, r_3 are described. Shew that the equations of the lines joining two centres of similitude of the three circles are given by

$$r_1 a\alpha \pm r_2 b\beta \pm r_3 c\gamma = 0.$$

15. The six centres of similitude of three circles lie three by three upon four straight lines.

16. Shew that the equation of the nine-point circle of the triangle of reference is of the form

$$k\beta\gamma = (b\beta + c\gamma - a\alpha)(\alpha\cos A - \beta\cos B - \gamma\cos C),$$

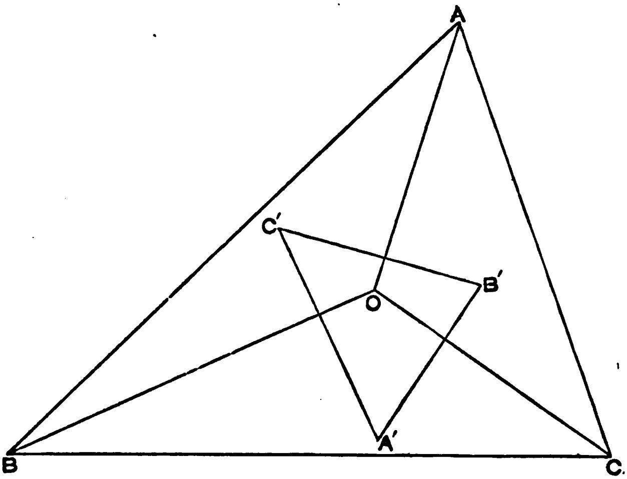
and determine k.

17. Shew that the equation of the nine-point circle is

$$(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = \frac{(a\alpha + b\beta + c\gamma)}{2}(\alpha\cos A + \beta\cos B + \gamma\cos C).$$

18. Find the conditions that the nine-point circle may coincide with the inscribed circle of the triangle.

19. Let $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ be the equations of three circles which touch each other mutually, then if l_1



be the length of the common tangent of S_2 and S_3, and if l_2 and l_3 are similar lengths, the equation of the circle touching S_1, S_2 and S_3 is

$$l_1\sqrt{S_1} + l_2\sqrt{S_2} + l_3\sqrt{S_3} = 0.$$

Let A, B, C be the centres of S_1, S_2, S_3; let r_1, r_2, r_3 be their radii. Take O the centre of the circle required, and join OA, OB, OC.

Let $A'B'C'$ be the triangle formed by the common tangents of the circle, and of S_1, S_2, S_3 respectively. Then taking $A'B'C'$ as triangle of reference, the equation of the circle required, being the inscribed circle of $A'B'C'$, is

$$\sqrt{(1+\cos A')\,\alpha}+\sqrt{(1+\cos B')\,\beta}+\sqrt{(1+\cos C')\,\gamma}=0,$$

or

$$\sqrt{(1-\cos BOC)\,\alpha}+\sqrt{(1-\cos COA)\,\beta}+\sqrt{(1-\cos AOB)\,\gamma}$$
$$=0\ldots\ldots\ldots\ldots(1).$$

Now by (IX. 1)
$$\left.\begin{aligned}S_1&=2OA\,.\,\alpha\\ S_2&=2OB\,.\,\beta\\ S_3&=2OC\,.\,\gamma\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\,(2),$$

and

$$1-\cos BOC=\frac{BC^2-(BO-OC)^2}{2OB\,.\,OC}=\frac{BC^2-(r_2-r_3)^2}{2OB\,.\,OC}\ldots..(3)$$
$$=\frac{l_1^2}{2OB\,.\,OC},$$

and two similar equations.

Substituting from (3) and (2) in (1) we obtain

$$l_1\sqrt{S_1}+l_2\sqrt{S_2}+l_3\sqrt{S_3}=0,$$

the equation of the circle required.

This method of proof is due to Mr J. Griffiths, M.A., Fellow of Jesus College, Oxford.

20. A conic touching the sides of the triangle of reference where the bisectors of the angles meet the opposite sides meets another conic touching the sides AB, AC

at B and C, and passing through the point of intersection of the bisectors of angles; shew that

$$\alpha + \beta + \gamma = 0$$

is a chord of intersection of the two conics.

21. A conic touches the sides AB, AC of the triangle of reference at B and C, and passes through O the centre of the inscribed circle. If from any point in the tangent at O perpendiculars are drawn on the sides, the one on BC is the arithmetic mean between the other two.

22. The equation of the parabola touching the sides AB, AC at B and C is $a^2\alpha^2 = 4bc\beta\gamma$.

23. Two parabolas are drawn, one touching AB, AC at B and C and the other BA, BC at A and C. If the tangent to the first at the remaining point of intersection O meet AC in M, and the tangent to the second at O meet BC in N, shew that

$$MC = \tfrac{2}{3} AC, \text{ and } NC = \tfrac{2}{3} BC.$$

24. Two parabolas are drawn, one touches AB, AC at B and C, the other AC, BC at A and B respectively; shew that the area of the common loop

$$= \tfrac{32}{81} \Delta \,.\, ABC.$$

25. Find the condition that the conic inscribed in the triangle of reference should be a parabola.

26. Find the conditions that the general equation of the second degree in trilinear co-ordinates should represent a circle.

27. Find the condition that the general equation of the second degree in trilinear co-ordinates should represent a parabola.

28. If $f(\alpha, \beta, \gamma) = 0$ denote the general equation of the second degree, the direction-cosines λ, μ, ν of the asymptotes of the conic represented by it are given by $f(\lambda, \mu, \nu) = 0$.

29. The condition that the general equation of the second degree in trilinear co-ordinates should represent an equilateral hyperbola is

$$A + B + C - 2F \cos A - 2G \cos B - 2H \cos C = 0.$$

30. If an equilateral hyperbola circumscribe a triangle, it will also pass through the centre of perpendiculars.

31. The condition that the conic inscribed in a triangle should be an equilateral hyperbola is

$$l^2 + m^2 + n^2 + 2lm \cos C + 2ln \cos B + 2mn \cos A = 0.$$

32. Shew that it is impossible to inscribe an equilateral hyperbola in an equilateral triangle.

33. The internal bisector of the angle ABC meets the bisectors of the angle ACB in P and P' respectively; the external bisector of the angle ABC meets the same lines in Q and Q' respectively; shew that a conic circumscribing PP', QQ' must be an equilateral hyperbola, with respect to which the triangle of reference is self-conjugate.

34. Prove that $l\alpha + m\beta + n\gamma = 0$ will represent an asymptote to an equilateral hyperbola circumscribing the triangle of reference, if

$$l(mc - nb)^2 \cos A + m(na - lc)^2 \cos B + n(lb - ma)^2 \cos C = 0,$$

and find the equation of the other asymptote.

35. Find the co-ordinates of the centre of the hyperbola in the preceding example, and shew that

$$\frac{\alpha}{l_1}+\frac{\beta}{m_1}+\frac{\gamma}{n_1}=0$$

is a diameter; where $l_1=\dfrac{l\,(m^2c^2-n^2b^2)}{c-b}$ and m_1 and n_1 have similar values.

36. If a hyperbola be such that the triangle of reference is self-conjugate with respect to it, and if

$$l\alpha+m\beta+n\gamma=0$$

be an asymptote, find the other asymptote, and the condition that the hyperbola should be rectangular.

XXII. ENVELOPES.

In the following examples the indeterminate constant will appear in the second degree; the condition that the quadratic in it should have equal roots determines the equation of the envelope.

1. Find the envelope of the straight line $y=mx+c$, when the difference of the squares of the perpendiculars on it from the points $(\pm a, 0)$ is given.

2. Find the envelope of a line that cuts off a triangle of constant area from two given straight lines.

3. A straight line cuts off intercepts from two given straight lines. Find the envelope when the sum of the intercepts is constant.

4. Two circles

$$x^2+y^2=a^2,$$
$$(x-a)^2+y^2=b^2,$$

cut off equal lengths on the variable straight line $y=mx+c$; find its envelope.

5. Find the envelope of the chord joining the extremities of a pair of conjugate diameters of an ellipse.

6. The chord of a circle whose centre is at C subtends a right angle at O. The envelope of the chord is a conic whose focus is at O and whose eccentricity

$$=\sqrt{\frac{OC^2}{2r^2-OC^2}},$$

r being the radius of the circle.

7. Find the envelope of

$$l\alpha+m\beta+n\gamma=0$$

if
$$\frac{a}{l}+\frac{b}{m}+\frac{c}{n}=0.$$

8. Find the envelope of a straight line, if the sum of the reciprocals of the perpendiculars on it from the vertices of a given triangle is zero.

9. Find the envelope of the line in (8), if the sum of the squares of two of the perpendiculars is equal to the square of the third.

10. If p, q, r be the perpendiculars from the vertices of the triangle of reference on a straight line, shew that it envelopes the circle circumscribing the triangle if

$$a\sqrt{p}+b\sqrt{q}+c\sqrt{r}=0.$$

XXIII.

[*The greater part of the following equations have been selected from University and Scholarship papers.*]

Trace the curves:

1. $3x^2 - 5xy + 3y^2 - x - y = 0.$
2. $4x^2 + 3x - 5y = 0.$
3. $2y^2 - 3xy - 2x^2 + y + 3x + 1 = 0.$
4. $x^2 + 4y^2 + 12y + 6x + 4xy + 9 = 0.$
5. $y^2 - 4xy - 5x^2 + 6y + 42x - 63 = 0.$
6. $9y^2 + 16x^2 - 18y + 32x - 119 = 0.$
7. $x^2 + xy - 6y^2 - 5x - 5y + 6 = 0.$
8. $y^2 + xy - 2x^2 + 3x - 1 = 0.$
9. $x^2 + y^2 - 2xy - 2x - 2y + 1 = 0.$
10. $5x^2 + 11xy + 2y^2 - 13x + 10y - 28 = 0.$
11. $2x^2 - xy + 6y^2 - 3x + 13y - 5 = 0.$
12. $3x^2 - 2xy + 3y^2 - 6x + 2y = 0.$
13. $2x^2 - xy - y^2 + 7x - y + 6 = 0.$
14. $y^2 - 2\sqrt{3}\,xy + 3x^2 + 5\sqrt{3}\,.\,x - 5y + 6 = 0.$
15. $\sqrt{x+a} + \sqrt{y+b} = \sqrt{2a}.$
16. $(3x - 2y + 4)(x + 5y - 7) = 4.$
17. $x^2 + y^2 = a^2$ (when the axes are oblique).

18. $\frac{1}{x}+\frac{1}{y}=2.$

19. $(ax-by+c)^2+(bx-ay+d)^2=(c^2+d^2)^2.$

20. $(x\cos\alpha+y\sin\alpha)^2-4m(x\cos\beta+y\sin\beta)+4m^2=0.$

21. $\cos^{-1}x+\cos^{-1}y=\frac{\pi}{3}.$

22. $xy=x+y.$

23. $y=\frac{a^2-x^2}{2a-x}.$

24. $\frac{x^2}{a^2}-2\cos c\,.\,\frac{xy}{ab}+\frac{y^2}{b^2}=\sin^2 c.$

25. $(a^2+b^2)(x^2+y^2)=(ax+by)^2.$

26. $(ax-by)^2-2cbx-2cay-c^2=0.$

27. $(y-x)^2=y+x+1.$

28. $\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=\left(\frac{x}{a}-1\right)\left(\frac{y}{b}-1\right).$

29. $r=a\sin\theta.$

30. $r(1+2\cos\theta)=3.$

31. $\frac{2}{r}=1+\cos\theta+\sin\theta.$

32. $r+\frac{1}{r}=2\cos\theta+2\sin\theta.$

33. $r=\sqrt{2}(r\cos\theta-2r\sin\theta+9).$

34. $\frac{1}{r}=1+\cos\theta+\sqrt{3}\sin\theta.$

35. $r = 2a\left(\dfrac{\cos\theta \pm \sqrt{\cos 2\theta}}{\sin^2\theta}\right).$

36. $r = 2a(\cos\theta + \sin\theta).$

37. $r = a\sec 45^\circ . \sec\dfrac{\theta - 45^\circ}{2}.$

38. $r = \sec^2\dfrac{\theta - \alpha}{2}.$

39. $(a\alpha + b\beta + c\gamma)^2 = (l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma).$

40. $\alpha^2 - \beta^2 = \gamma^2.$

41. $\sqrt{3\alpha} + \sqrt{4\beta} + \sqrt{5\gamma} = 0.$

42. $(-2\alpha)^{\frac{1}{2}} + (\sqrt{2}\beta)^{\frac{1}{2}} + \{(\sqrt{6} - \sqrt{2})\gamma\}^{\frac{1}{2}} = 0.$

43. $\sqrt{-\alpha\sin A} + \sqrt{\beta\sin B} + \sqrt{\gamma\sin C} = 0.$

44. $ab\alpha\beta + bc\beta\gamma + ca\gamma\alpha = 0.$

45. $\alpha + \beta + \gamma = 2\sqrt{\alpha\beta + \beta\gamma + \gamma\alpha}.$

46. $\beta\gamma = \alpha^2\cos A.$

47. $A\alpha^2 + B\beta^2 = (A + B)\gamma^2.$

48. $\sin(B - C)\beta\gamma + \sin(C - A)\gamma\alpha + \sin(A - B)\alpha\beta = 0.$

49. $a\sqrt{\sin(B - C) . \alpha} + b\sqrt{\sin(C - A) . \beta}$
$+ c\sqrt{\sin(A - B) . \gamma} = 0.$

50. $\alpha\beta = k\gamma(a\alpha + b\beta + c\gamma).$

Can this last equation ever represent an equilateral hyperbola?

ANSWERS, HINTS, &c.

I. CO-ORDINATES, &c.

1. 65.

2. 15; $3\sqrt{13}$.

5. i. 41.

   ii. 65.

7. $9\sqrt{3}$.

8. 8·375.

10. Let (x_1y_1), (x_2y_2), (x_3y_3), be the co-ordinates of the vertices, the co-ordinates of the point are $\frac{x_1+x_2+x_3}{3}$, $\frac{y_1+y_2+y_3}{3}$.

11 and 12. $\frac{x_1+x_2+x_3+x_4}{4}$, $\frac{y_1+y_2+y_3+y_4}{4}$.

13. Of $C\left(\frac{65\sqrt{3}}{2}, \frac{65}{2}\right)$;

of $B\ \{\frac{28}{65}(56\sqrt{3}+33), \pm\frac{28}{65}(56-33\sqrt{3})\}$.

15. The radius being $\frac{a}{2\sin A}$ if a is the side opposite the origin, $\frac{\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}}{2\sin(\theta_2-\theta_1)}$ is the value required.

II. TRANSFORMATION OF CO-ORDINATES.

1. $(x^2 - y^2)\sqrt{3} + 2xy + (4 + 6\sqrt{3})\,x + (4\sqrt{3} - 6)\,y + 24 = 0.$

$x^2 - y^2 - 2\sqrt{3}xy + 2\sqrt{3}\,(2 - \sqrt{3})\,x - 2\,(2 + 3\sqrt{3})y = 10.$

2. The angle between the former axes is 120°, that between the old and new axes of x is 60°. Draw a figure.

3. $(x + my)^2 + 8a\,(1 + m^2)^{\frac{3}{2}}y = 0.$

4. $\dfrac{x^2}{m^2} + y^2\left(\dfrac{l^2\cos^2\omega + m^2}{l^2m^2\sin^2\omega}\right) - \dfrac{2xy\cot\omega}{m^2} - \dfrac{2y}{l\sin\omega} = 0.$

III. THE STRAIGHT LINE. RECTANGULAR CO-ORDINATES.

1. $5y + 9x + \frac{17}{4} = 0.$ To obtain the equation of a straight line joining the intersections of $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$; $a_3x + b_3y = c_3$, $a_4x + b_4y = c_4$. The line must be of the form

$$a_1x + b_1y - c_1 + k\,(a_2x + b_2y - c_2) = 0 \ldots\ldots\ldots\ldots (1)$$

and also of form

$$a_3x + b_3y - c_3 + l\,(a_4x + by_4y - c_4) = 0 \ldots\ldots\ldots (2),$$

whence
$$\frac{a_1 + ka_2}{a_3 + la_4} = \frac{b_1 + kb_2}{b_3 + lb_4} = \frac{c_1 + kc_2}{c_3 + lc_4} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

since the two equations (1) and (2) represent the same line. The equations (3) determine k and l; of course it is only necessary to find one of them.

2. $2y - 7x + 25 = 0.$

3. $\dfrac{38}{5\sqrt{10}}.$

4. $2y - x + 17 = 0.$

5. $\dfrac{4}{\sqrt{5}}.$

6. $\tan^{-1}(-\frac{11}{2})$.

7. $(\frac{88}{41}, \frac{110}{41})$.

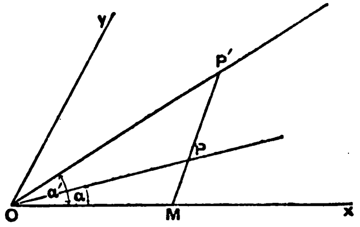
8. $4y + 3x - 5a = 0$.

9. $\frac{4}{11}\sqrt{17}$.

10. $(\frac{4}{3}, -4)$; $\tan^{-1}(-\frac{3}{2})$.

11. $\tan^{-1}\dfrac{4m^2m'^2}{m'^4 - m^4}$.

14. Let Ox, Oy be the axes, let OP, OP' denote $y = mx$, $y = m'x$ respectively; call POM α, $P'OM$ α';



Then
$$\frac{PM}{OM} = m = \frac{\sin\alpha}{\sin(\omega - \alpha)},$$

whence
$$\cot\alpha\sin\omega - \cos\omega = \frac{1}{m} \quad\text{.......................(1)};$$

similarly
$$\cot\alpha'\sin\omega - \cos\omega = \frac{1}{m'} \quad\text{.......................(2)};$$

from (1) and (2) form $\cot\alpha - \cot\alpha'$ and $\cot\alpha\cot\alpha'$ and substitute in

$$\tan\theta = \frac{\cot\alpha - \cot\alpha'}{1 + \cot\alpha\cot\alpha'}, \text{ whence}$$

$$\tan\theta = \frac{(m' - m)\sin\omega}{1 + mm' - (m + m')\cos\omega}.$$

Note. This finds the angle between $y=mx+c$ and $y=m'x+c'$, since the angle between these lines is the same as the angle contained by $y=mx$, $y=m'x$, parallels to them drawn through the origin.

17. $x\left(\dfrac{1}{a}-\dfrac{1}{a'}\right)+y\left(\dfrac{1}{b}-\dfrac{1}{b'}\right)=0.$

18. $\dfrac{x}{n}+\dfrac{y}{m}=\dfrac{l}{m+n}.$

20. Let $OA=l$, $OA'=m$.

Then the equation of AB' is $\dfrac{x}{l}+\dfrac{y}{l\sec\omega}=1$,

similarly that of $A'B$ is $\dfrac{x}{m}+\dfrac{y}{m\cos\omega}=1$;

making these lines pass through (a, b) we have two equations to find l and m, whence

$$AA'=m-l=\frac{b\sin^2\omega}{\cos\omega},$$

$$BB'=l\sec\omega-m\cos\omega=\frac{a\sin^2\omega}{\cos\omega}.$$

23. (1) $3y+x+6=0$, $2y-x+6=0$; 45°.

(2) $y=(2+\sqrt{3})\,x-4$,
$y=(2-\sqrt{3})\,x-5$;
60°.

(3) $y=\dfrac{b}{a}x-\dfrac{c}{a}$,

$y=-\dfrac{a}{b}x+\dfrac{c}{b}$; 90°.

24. A straight line through the origin.

25. The co-ordinates of the point of intersection are (a, b).

27. If (x_1, y_1), (x_2, y_2), (x_3, y_3) be the co-ordinates of the vertices A, B, C, the line bisecting BC at right angles is

$$x(x_3 - x_2) + y(y_3 - y_2) = \frac{x_3^2 + y_3^2 - x_2^2 - y_2^2}{2}.$$

28.

$$x = \frac{(x_1^2 + y_1^2)(y_3 - y_2) + (x_2^2 + y_2^2)(y_1 - y_3) + (x_3^2 + y_3^2)(y_2 - y_1)}{2\{(x_3 - x_2)(y_2 - y_1) - (y_3 - y_2)(x_2 - x_1)\}},$$

$$y = \frac{(x_1^2 + y_1^2)(x_2 - x_3) + (x_2^2 + y_2^2)(x_3 - x_1) + (x_3^2 + y_3^2)(x_1 - x_2)}{2\{(x_3 - x_2)(y_2 - y_1) - (y_3 - y_2)(x_2 - x_1)\}}.$$

31. See I. 11.

The locus is $\frac{h}{4x - a} + \frac{k}{4y - b} = 1$ where h, k, a, b are connected by the relation

$$\frac{h}{a} + \frac{k}{b} = 1.$$

32. (1) $\frac{a^2 \sin^2 \alpha}{\cos \alpha (\sin 2\alpha + \cos 2\alpha)}$; (2) $\frac{a^2}{2}$.

33. $\frac{ab}{24}$.

35. This depends on the identity

$$a^4 + 4 = (a^2 + 2a + 2)(a^2 - 2a + 2).$$

37. $x = al^2$, $y = b(1 - l)^2$.

38. $y - b = \frac{m + m'}{1 - mm'}(x - a)$,

$$y - b = \frac{mm' - 1}{m + m'}(x - a).$$

IV. PAIRS OF STRAIGHT LINES.

6. Let $y^2 + \frac{2hxy}{b} + \frac{ax^2}{b} = (y - mx)(y - m'x)$, then since the bisectors are the locus from which equal perpendiculars can be drawn, the locus consists of the product of

$$\frac{y - mx}{\sqrt{1 + m^2}} = \pm \frac{y - m'x}{\sqrt{1 + m'^2}},$$

or $$(y - mx)^2 (1 + m'^2) - (y - m'x)^2 (1 + m^2) = 0;$$

multiply out, and substitute for $m + m'$, mm', $m' - m$ &c., their values.

V. THE STRAIGHT LINE REFERRED TO POLAR CO-ORDINATES.

2. $$\theta = \frac{\theta_1 + \theta_2}{2},\ \theta = \frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2};\ \frac{2r_1r_2}{r_1 + r_2} \cos\left(\frac{\theta_2 - \theta_1}{2}\right),$$
$$\frac{2r_1r_2}{r_1 - r_2} \sin\left(\frac{\theta_2 - \theta_1}{2}\right).$$

3. $$\frac{\cos\theta_1 + \cos\theta_2}{r} = \cos\theta\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

4. $$r = \frac{r_1r_2 \sin(\theta_2 - \theta_1)}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}},$$
$$\theta = \tan^{-1}\frac{r_1\cos\theta_1 - r_2\cos\theta_2}{r_2\sin\theta_2 - r_1\sin\theta_1}$$

5. $r\sin\left(\theta + \frac{\pi}{4}\right) = a$, a straight line.

6. Take O as pole. The locus is the straight line

$$r = OC \cdot \frac{\sin\left(OCA + \frac{ACB}{2}\right)}{\sin\left(\theta + OCA + \frac{ACB}{2}\right)}.$$

7. Take $O'O$ as initial line, O' as pole. Let the inclination of the transversal to OO' be θ and call aOO', a, &c. Then the locus is the straight line

$$\frac{n}{r}=\frac{1}{a}\left(\frac{\sin(\alpha+\theta)}{\sin\alpha}+\frac{\sin(\beta+\theta)}{\sin\beta}+\dots\right),$$

where $a=OO'$.

VI. THE STRAIGHT LINE REFERRED TO TRILINEAR CO-ORDINATES.

2. Take $\alpha_1, \beta_1, \gamma_1$ to be a point on $l\alpha+m\beta+n\gamma=0$. Then this straight line may also be written

$$\frac{\alpha-\alpha_1}{\cos\theta}=\frac{\beta-\beta_1}{\cos\phi}=\frac{\gamma-\gamma_1}{\cos\psi}=r,$$

whence $(l\cos\theta+m\cos\phi+n\cos\psi)\,r$

$$=l\alpha+m\beta+n\gamma-(l\alpha_1+m\beta_1+n\gamma_1)=0;$$

since (α, β, γ) $(\alpha_1, \beta_1, \gamma_1)$ both lie on $l\alpha+m\beta+n\gamma=0$.

3. If θ, ϕ, ψ be the angles which $l\alpha+m\beta+n\gamma=0$ makes with the perpendiculars on the sides, $\frac{\alpha_1-\alpha}{\sin\theta}=\frac{\beta_1-\beta}{\sin\phi}=\frac{\gamma_1-\gamma}{\sin\psi}$ will represent a line through $(\alpha_1, \beta_1, \gamma_1)$ perpendicular to the given line. Let p be the length of the perpendicular required, then if (α, β, γ) be the co-ordinates of the foot of the perpendicular from $(\alpha_1, \beta_1, \gamma_1)$ on the given line

$$\frac{\alpha_1-\alpha}{\sin\theta}=\frac{\beta_1-\beta}{\sin\phi}=\frac{\gamma_1-\gamma}{\sin\psi}=p,$$

whence

$$p\{l\sin\theta+m\sin\phi+n\sin\psi\}=l\alpha_1+m\beta_1+n\gamma_1-(l\alpha+m\beta+n\gamma);$$

$$\therefore\ p=\frac{l\alpha_1+m\beta_1+n\gamma_1}{l\sin\theta+m\sin\phi+n\sin\psi},\ \text{since } l\alpha+m\beta+n\gamma=0.$$

Now

$$(l \sin \theta + m \sin \phi + n \sin \psi)^2 = (l \sin \theta + m \sin \phi + n \sin \psi)^2$$

$+ (l \cos \theta + m \cos \phi + n \cos \psi)^2$ since $l \cos \theta + m \cos \phi + n \cos \psi = 0$
$= l^2 + m^2 + n^2 + 2mn \cos (\phi - \psi) + 2nl \cos (\psi - \theta) + 2lm \cos (\theta - \phi)$,
but $\phi - \psi$ is the supplement of the angle A of the triangle of reference, and similarly for the others;

$$\therefore \quad l \sin \theta + m \sin \phi + n \sin \psi$$

$$= \sqrt{(l^2 + m^2 + n^2 - 2mn \cos A - 2ln \cos B - 2lm \cos C)}.$$

4. Write out the equations of the lines in full length in Cartesian co-ordinates in the form

$$(l \cos \alpha + m \cos \beta + n \cos \gamma)\, x + (l \sin \alpha + m \sin \beta + n \sin \gamma)\, y + \ldots = 0,$$

$$(l' \cos \alpha + m' \cos \beta + n' \cos \gamma)\, x + (l' \sin \alpha + m' \sin \beta + n' \sin \gamma)\, y + \ldots = 0,$$

then apply the condition of perpendicularity and finally substitute for $\alpha - \beta$, $180^\circ - C$, &c.

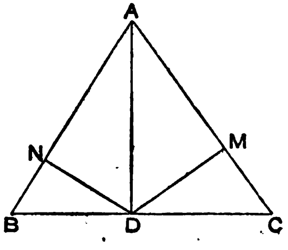
This is the method used in Salmon's *Conic Sections*.

5. i. To find the bisector of the angle BAC. Since it passes through A it is of the form

$$\beta - k\gamma = 0, \therefore k = \frac{\beta}{\gamma} = \frac{DM}{DN} = 1$$

in this case, $\therefore$ the equations of the bisectors are

$$\left.\begin{array}{l} \beta - \gamma = 0 \\ \gamma - \alpha = 0 \\ \alpha - \beta = 0 \end{array}\right\}.$$



ii. By a similar investigation to i. $k = \frac{c}{b}$,

$\therefore$ the lines are

$$\left.\begin{aligned} a\alpha - b\beta = 0 \\ b\beta - c\gamma = 0 \\ c\gamma - a\alpha = 0 \end{aligned}\right\}.$$

iii. $\alpha \cos A - \beta \cos B = 0$, $\beta \cos B - \gamma \cos C = 0$, $\gamma \cos C - \alpha \cos A = 0$.

iv. Make the line $l\alpha + m\beta + n\gamma = 0$ pass through the middle point of BC and make it perpendicular to $\alpha = 0$ by example (4), or thus; since it passes through the intersection of $\beta \sin B - \gamma \sin C = 0$ and of $\alpha = 0$, it may be written $\beta \sin B - \gamma \sin C + k\alpha = 0$. Substituting now in the condition of perpendicularity, we obtain k.

Either method gives $\beta \sin B - \gamma \sin C + \alpha \sin (B - C) = 0$ and similar equations for the other two.

(6) i. $1 : 1 : 1$.

ii. $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

iii. Sec A : sec B : sec C.

iv. Cos A : cos B : cos C.

The absolute values of these co-ordinates may be found thus: in iv. for example let ($k \cos A$, $k \cos B$, $k \cos C$) be the absolute values required, substituting in

$$a\alpha + b\beta + c\gamma = 2\Delta \text{ we obtain } k = \frac{2\Delta}{a \cos A + b \cos B + c \cos C}$$

$$= \frac{a}{2 \sin A} = R.$$

7. i. $\alpha + \beta - \gamma = 0$, $\beta + \gamma - \alpha = 0$, $\gamma + \alpha - \beta = 0$.

ii. $\alpha \cos A + \beta \cos B - \gamma \cos C = 0$,
$\beta \cos B + \gamma \cos C - \alpha \cos A = 0$,
$\gamma \cos C + \alpha \cos A - \beta \cos B = 0$.

8. $$a\alpha \cos A \sin (B-C) + b\beta \cos B \sin (C-A) + c\gamma \cos C \sin (A-B) = 0.$$

10. Let $l\alpha + m\beta + n\gamma = 0$ represent a straight line. The perpendicular on it from A is given by

$$p = \frac{\frac{2\Delta}{a} . l}{\sqrt{l^2 + m^2 + n^2 - 2mn \cos A - 2ln \cos B - 2lm \cos C}},$$

whence $l : m : n :: pa : qb : rc$.

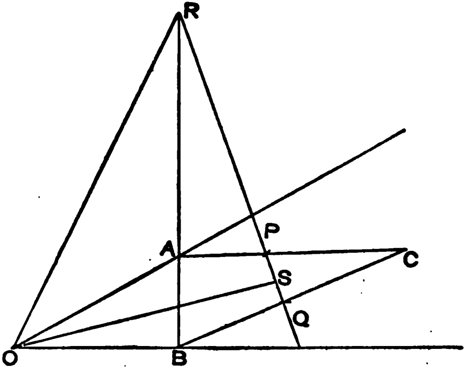
12. i. $\operatorname{cosec} A (\cos B - \cos C) : \operatorname{cosec} B (\cos C - \cos A) : \operatorname{cosec} C (\cos A - \cos B)$.

ii. $\cos A \sin (B - C) : \cos B \sin (C - A) : \cos C \sin (A - B)$.

13. $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$.

14. The line $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0$.

15. Take OR as $\alpha = 0$, PQR as $\gamma = 0$, and a line OS as $\beta = 0$, where OS is so placed as to make the equations OA and



OB $\alpha-\beta=0$ and $\alpha+\beta=0$ respectively. Let *RAB* be $\lambda\alpha=\gamma$, then the equation of *AC* is

$$(\lambda\alpha-\gamma)(\alpha_1-\beta_1)-\lambda\alpha_1(\alpha-\beta)=0\ldots\ldots\ldots\ldots\ldots(1),$$

of *BC* is $$(\lambda\alpha-\gamma)(\alpha_2+\beta_2)-\lambda\alpha_2(\alpha+\beta)=0\ldots\ldots\ldots\ldots\ldots(2),$$

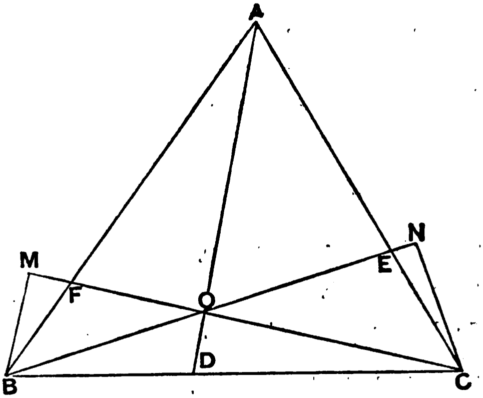
if $(\alpha_1, \beta_1, 0)$ $(\alpha_2, \beta_2, 0)$ are the co-ordinates of *P* and *Q* respectively; eliminating λ between (1) and (2) we obtain

$$\alpha\left\{\frac{\beta_1}{\alpha_1-\beta_1}+\frac{\beta_2}{\alpha_2+\beta_2}\right\}=\beta\left\{\frac{\alpha_1}{\alpha_1-\beta_1}+\frac{\alpha_2}{\alpha_2+\beta_2}\right\},$$

a straight line through the point *O*, as the locus.

16. Take $m\beta-n\gamma=0$, $n\gamma-l\alpha=0$, $l\alpha-m\beta=0$; as the straight lines meeting at *O*. Then *L*, *M*, *N*, lie on $l\alpha+m\beta+n\gamma=0$.

17. Take *AD* of the form $m\beta-n\gamma=0$ and similar equations for *BE* and *CF*. From *B* and *C* draw *BM* and *CN* perpendicular to *CO* and *BO* respectively.



Then

$$\sin BOC=\frac{BM}{BO}=\frac{CN}{CO}, \text{ whence}$$

$$2\triangle BOC\,.\,\sin BOC=BM\,.\,CN\ldots\ldots\ldots\ldots\ldots(1).$$

Now BM and CN are perpendiculars from the vertices and $2\ BOC = a \cdot \frac{k}{l}$ where $\frac{k}{l}$ is the perpendicular from O on BC. Substituting in (1) we obtain an expression for

$$\frac{\sin BOC}{\sqrt{m^2 + n^2 - 2mn \cos A}}$$

which is not altered by cyclic transformation.

23. Take a point O (α, β, γ) within the triangle, then if $(0, \beta_1, \gamma_1)$ are the co-ordinates of the foot of the perpendicular on BC; $\beta_1 = \alpha \cos C + \beta$, $\gamma_1 = \alpha \cos B + \gamma$, and similar equations for the others. Should O lie without the triangle, but within the angle BAC, the sign of α would become negative.

25. $\frac{\sin (A - \theta)}{\sin \theta} = k_1$ &c., hence we can establish the relation

$$\frac{\sin (A - \theta + \psi)}{\cos \theta \cos \psi} = \frac{\sin A \cdot k_1 k_2 + \sin B \cdot k_1 + \sin C}{(k_1 + \cos A)(k_2 + \cos C)}.$$

26. Let the triangle formed by the intersections be $A'B'C'$. Let p be the length of the perpendicular from A' on BC. Let q and r be similar perpendiculars. Then $2\Delta' = 2\Delta - ap - bq - cr$.

VII. THE CIRCLE REFERRED TO CARTESIAN CO-ORDINATES.

1. The diameter is $y + 3x = 0$.
2. $3x^2 + 3y^2 = 58x$.
3. $\frac{7}{6}$; $\frac{1}{2}\sqrt{29}$; $(0, \frac{7}{6})$, $(1, -\frac{3}{2})$.
4. $\frac{\sqrt{11}}{3}$, $(-\frac{2}{3}, \frac{2}{3})$, $\frac{1}{3}(-2 \pm \sqrt{7})$.
5. $(x - \frac{47}{17})^2 + (y - \frac{36}{17})^2 = \frac{7585}{289}$.
6. $(x - \frac{9}{7})^2 + (y - \frac{1}{14})^2 = \frac{9425}{196}$.
8. b.

10. $8y - 6x = 25.$

11. 2; $5y = 2(9 \pm \sqrt{41})\,x.$

12. $\tan^{-1}(\frac{3}{4}).$

13. $x^2 + y^2 + 6y - 2x + \frac{49}{5} = 0.$

14. $(1, 7)$; $(-\frac{1}{2}, -\frac{1}{2})$; $\frac{1}{2}\sqrt{234}.$

15. $\dfrac{-2 \pm \sqrt{2}}{2}, \ \dfrac{-2 \pm 3\sqrt{2}}{2}.$

16. $\tan^{-1}\left(\pm \dfrac{3}{\sqrt{7}}\right).$

17. If $$l = \sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2}$$
the conditions are $y_1(x_2 - x_1) - x_1(y_2 - y_1) = lr$ and two similar ones.

19. Find the condition that the line joining the current point (x, y) to the point (x_1, y_1) shall be perpendicular to the line joining (x, y) to (x_2, y_2).

23. $\theta = 2\tan^{-1}\dfrac{a}{b}$. Draw a figure.

27. Take the common tangent as axis of y, the line joining the centres as axis of x.

28. $(x - a)^2 + y^2 = \dfrac{a^2}{4}.$

29. $\cos^{-1}\left\{\dfrac{(h-\alpha)^2 + (k-\beta)^2}{r^2}\right\}^{-\frac{1}{2}}.$

30. Let l be the length of the chords. The locus is a concentric circle $x^2 + y^2 = \dfrac{4r^4}{4r^2 - l^2}$. Explain this result when $l = 2r$.

31. If d be the distance between (h, k) and (h', k') the condition is $rd = hk' \sim h'k$; r being the radius of the circle.

34. Take the diameter as axis of y and a perpendicular to it at the centre as axis of x.

36. Two circles are said to cut orthogonally when their tangents at the points of intersection are at right angles. It thus follows that the radii to those points must be at right angles. The condition algebraically expressed is $c^2 = r^2 + r'^2$ where c is the distance between the centres and r and r' are the radii of the circles; applying this method to the example, taking the circle to be $(x-\alpha)^2 + (y-\beta)^2 = r^2$, we have

$$\alpha^2 + \beta^2 = r^2 + a^2,$$
$$(\alpha - b)^2 + \beta^2 = r^2 + a^2,$$
$$\alpha^2 + (\beta - c)^2 = r^2 + a^2,$$

whence the circle is found to be

$$x^2 + y^2 - bx - cy + a^2 = 0.$$

37. A circle.

41. $x = \frac{R}{2} \sin (B - C), \quad y = \frac{R}{2} \cos (B - C).$

The radius of the nine-point circle is equal to half the radius of the circumscribing circle of the triangle.

42. If R be the radius of the circumscribing circle the distance is $\frac{R}{2} \sqrt{1 - 8 \cos A \cos B \cos C}$.

43. $t_1^2 = \frac{bc}{2} \cdot \cos A.$

45. $x^2 + y^2 - 2c \cos B x - 2c \cos B \cot C y$

$$= \frac{c^2}{\sin^2 C} \{2 \cos A \cos B \cos C - \sin A \sin C \cos B\}.$$

Take B as origin and BC and the perpendicular to it as axes of x and y.

48. $x^2 + 2xy \cos \omega + y^2 - 2b (x + y) + b^2 = 0.$

51. $x^2 + y^2 = a^2 + l^2$ where a is the radius of the circle, l the constant length measured off.

52. The length is $r \sin \omega$ where r is the radius and ω the angle between the diameters.

Let C be the point from which the perpendiculars CA, CB are drawn, O the centre of the circle,

then $$AB^2 = AC^2 + BC^2 + 2AC \cdot BC \cdot \cos \omega,$$
and $$AC = r \sin AOC, \text{ \&c.}$$

VIII. The Circle referred to Polar Co-ordinates.

4. Let c be the distance of O from C the centre of the circle, a its radius. Then $\dfrac{OP + OQ}{2} = c \cos \theta$ and $OP \cdot OQ$ is known.

The locus is found to be $\dfrac{1}{r^2} = c^2 (3 \cos^2 \theta - \sin^2 \theta) + a^2$.

7. $$\left.\begin{aligned} l^2 &= r^2 + r_1^2 - 2rr_1 \cos (\theta - \theta_1) \\ a^2 &= r_1^2 + c^2 - 2r_1 c \cos (\theta_1 - a) \end{aligned}\right\}.$$

8. $r = c$.

IX. The Radical Axis of two or more Circles.

3. $x^2 + y^2 - 2hy + b^2 = 0$.

4. $(x - a)(x + b) + y^2 = a\sqrt{(x + b)^2 + y^2}$ where a and b are the radii of the circles, the common tangent being the axis of y. See VII. 25.

X. The Ellipse referred to its centre as origin.

3. $\pm \dfrac{a^2}{\sqrt{a^2 + b^2}}, \quad \pm \dfrac{b^2}{\sqrt{a^2 + b^2}}$.

4. $\pm\frac{2ae}{1+e^2}$ where e is the eccentricity of the ellipse.

6. The equation of the chord joining two points (x_1, y_1) (x_2, y_2) on an ellipse is

$$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1,$$

for this reduces to that of a straight line, and vanishes when $x=x_1$, $y=y_1$; or when $x=x_2$, $y=y_2$. Reducing and substituting for (x_1, y_1) (x_2, y_2) the eccentric notation we get the required form.

7. $2\sin\frac{\theta-\phi}{2}\left\{a^2\sin^2\frac{\theta+\phi}{2}+b^2\cos^2\frac{\theta+\phi}{2}\right\}^{\frac{1}{2}}$.

8. $\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$; $(-a\sin\theta, b\cos\theta)$.

9. $\dfrac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}, \dfrac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}$.

10. In (6) $\cos\theta+\cos\phi=\frac{2x_1}{a}$, $\sin\theta+\sin\phi=\frac{2y_1}{b}$.

11. An ellipse with axes parallel to those of the given one.

14. Take the equations of the chord referred to its middle point. The conditions that it and $\frac{x}{p}+\frac{y}{q}=1$ should be at right angles at (x_1, y_1), the middle point, give

$$\left.\begin{aligned}\frac{x_1}{a^2p}+\frac{y_1}{b^2q}&=0\\ \frac{x_1}{p}+\frac{y_1}{q}&=1\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

which in general only find one point and therefore only one chord. The point is found to be $\left(\frac{a^2p}{a^2-b^2}, \frac{b^2q}{b^2-a^2}\right)$.

15. The equations are the expressions of the conditions (1) That the line joining the centre of the circle to the middle point of the second chord should be perpendicular to that chord. (2) That the two chords are equally inclined to the axis of the ellipse, which follows from Euclid III. 35 and the theorem that the ratio of the rectangles contained by the segments of two intersecting chords of a conic is the same as that of the squares of the two parallel diameters (see Note in XX.).

19. $x^2+y^2=a^2+b^2$.

21. $\frac{2aby_1}{b^2-y_1^2}\sqrt{S}$, $\frac{2abx_1}{a^2-x_1^2}\sqrt{S}$; where $S=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$.

25. Make use of the fact that (x_1, y_1) must lie on

$$x^2+y^2=a^2+b^2. \quad \text{See (19).}$$

27. (1) $x+a\,.\,\frac{a^2+b^2}{a^2-b^2}=0$.

(2) $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$.

See example (26). The second result may be obtained by the more advanced student as follows:

The equation of the pair of lines joining the intersections of the polar of (x_1, y_1) with the ellipse to the centre is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2.$$

The condition that these should be at right angles, viz. that the sum of the coefficients of x^2 and y^2 should vanish, gives

$$\frac{1}{a^2}-\frac{x_1^2}{a^4}+\frac{1}{b^2}-\frac{y_1^2}{b^4}=0,$$

removing the suffixes we get the locus required.

28. $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{k^2}$.

30. $x^2+y^2-\frac{a^2-2b^2}{b}y=a^2-b^2$; $y=\frac{a^2-b^2}{b}$.

32. See (9) and (10) of this section and make use of the fact that CT bisects PQ.

36. Apply x. 8 and vii. 19 whence we obtain as the locus $2(x^2+y^2)^2=a^2x^2+b^2y^2$.

37. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

38. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{9}$.

39. Let the vertex C be (x, y), let p be the length of the perpendicular drawn from C on PQ. Let (x_1, y_1) be the middle point of PQ and α the angle it makes with the axis of y.

Then $x=x_1+p\cos\alpha=x_1+\frac{\sqrt{3}}{2}PQ\,.\cos\alpha$ .... (1).

Now x_1 and $PQ\cos\alpha$ are easily expressed in terms of the co-ordinates of P and Q. A similar equation to (1) is obtained for y. The locus is

$$\left(\frac{x}{a+\sqrt{3}b}\right)^2+\left(\frac{y}{b+\sqrt{3}a}\right)^2=\frac{1}{2}.$$

42. Find the equation of the circle described on the chord as diameter, then put $y=0$ in the same.

43. The chords, joining the points in pairs, must make equal angles with the major axis.

44. $ax\sin\theta-by\cos\theta=(a^2-b^2)\sin\theta\cos\theta$.

47. Eliminate x and y from the three equations

$$ax\sin\alpha-by\cos\alpha=(a^2-b^2)\sin\alpha\cos\alpha,$$

and two similar equations in β and γ.

49. By substituting for $\sin\theta$ and $\cos\theta$ in terms of $\tan\theta$, obtain from the equation in (44) the equation of the normal; which is of the fourth degree in $\tan\theta$. Then shew that if $\tan\alpha$, $\tan\beta$, $\tan\gamma$, $\tan\delta$ are the roots

$$1-(\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha+\tan\alpha\tan\delta$$
$$+\tan\beta\tan\delta+\tan\gamma\tan\delta)+\tan\alpha\tan\beta\tan\gamma\tan\delta=0,$$

which makes $\tan(\alpha+\beta+\gamma+\delta)$ infinite.

51. $x^2-2xy\cot\phi-y^2=a^2-b^2$.

55. $2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$.

56. Take the equation of the ellipse $\frac{x^2}{b'^2}+\frac{y^2}{a'^2}=1$, transform first to the point $(0, a')$ as origin, then from axes inclined at an angle ω, to rectangular axes, retaining the axis of x. Then the tangent will be the new axis of x, the normal the new axis of y; $\omega=\sin^{-1}\frac{ab}{a'b'}$.

57. The length at P is $\frac{2b'^3ab}{-a^2b^2+(a^2+b^2)b'^2}$;

the length at D is $\frac{2a'^3ab}{-a^2b^2+(a^2+b^2)a'^2}$.

60. An ellipse.

62. Let p' be the perpendicular from the other focus, then

$$\frac{p}{SP}=\frac{p'}{S'P}=\frac{b}{\sqrt{SP.S'P}}.$$

67. $x=-x_1$, $y=\frac{x_1^2-a^2e^2}{y_1}$.

70. If $BC=2a$ the locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ referred to OC, OMN as axes.

71. Let α, β, γ be the vertices of the triangles, then

$$\left.\begin{aligned}\cos\alpha+\cos\beta+\cos\gamma&=\frac{e}{3}\\ \sin\alpha+\sin\beta+\sin\gamma&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1),$$

and $\cos^2\dfrac{\alpha-\beta}{2}=\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}$, and similar relations. Square the equations (1) and add.

73. Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$$

intersect at (x_1, y_1); we have then

$$x_1^2\left(\frac{1}{a^2}-\frac{1}{a'^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{b'^2}\right)=0\ldots\ldots\ldots(1);$$

applying the condition $a^2-a'^2=b^2-b'^2$, (1) reduces to

$$\frac{x_1^2}{a^2a'^2}+\frac{y_1^2}{b^2b'^2}=0,$$

the condition that the tangents at the point of intersection should be at right angles.

74. Let $\dfrac{x^2}{a^2-\lambda}+\dfrac{y^2}{b^2-\lambda}=1$ be a conic through (x_1, y_1). We have to determine λ from the quadratic

$$\lambda^2+\lambda(x_1^2+y_1^2-a^2-b^2)+a^2b^2-b^2x_1^2-a^2y_1^2=0 \;\ldots\ldots\; (1):$$

this gives us two values of λ; therefore two conics can be described confocal with the others and passing through (x_1, y_1); again, in (1) put $\lambda=-\infty$, b^2, a^2 successively, when the sign of (1) will be $+$, $-$, $+$ respectively, shewing that one value of λ is less than b^2, the second greater than b^2 and less than a^2. The first value of λ will make the conic an ellipse, the second a hyperbola.

75. If $\sqrt{a^2-b^2}$ be the distance between the centre and a focus, the locus is

$$\frac{x}{m}+y+\frac{a^2-b^2}{c}=0.$$

76. Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1$$

be the two conics, the locus is

$$x^2+y^2=a^2+b^2-\lambda^2.$$

Use the form of tangent $x\cos\alpha+y\sin\alpha=p$.

77. A circle touching the perpendicular drawn to the axis of x at the point from which the tangents are drawn.

78. Take

$$\frac{x^2}{a_1^2}+\frac{y^2}{a_1^2-\lambda^2}=1,\ \frac{x^2}{a_2^2}+\frac{y^2}{a_2^2-\lambda^2}=1$$

as the confocal conics.

Then if P be (x_1, y_1), Q be (x_2, y_2),

$$\frac{xx_1}{a_1^2}+\frac{yy_1}{a_1^2-\lambda^2}=1,$$

$$\frac{xx_2}{a_2^2}+\frac{yy_2}{a_2^2-\lambda^2}=1$$

must be identical. This gives x_2 and y_2 in terms of x_1 and y_1, whence $(x_2-x_1)^2+(y_2-y_1)^2$ can be formed.

XI. THE ELLIPSE REFERRED TO POLAR CO-ORDINATES AND TO THE FOCUS AS POLE.

1. A conic having the same focus,

$$\frac{l\sec\alpha}{r}=1+e\sec\alpha\cos\theta.$$

2. The ellipse $\dfrac{l\cos^2\alpha}{r} = 1 + e\cos\alpha\cos\theta$.

3. Assume the equation to be $\dfrac{l}{r} = A\cos\theta + B\sin\theta$. Since the normal is perpendicular to the tangent

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta,$$

we have $$A(e + \cos\alpha) + B\sin\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again since the normal passes through α

$$1 + e\cos\alpha = A\cos\alpha + B\sin\alpha = -Ae \text{ by } (1).$$

$$\therefore\ A = -\frac{1 + e\cos\alpha}{e},\quad B = \frac{(1 + e\cos\alpha)(e + \cos\alpha)}{e\sin\alpha}.$$

Substituting for A and B we obtain the required result.

10. $\dfrac{l^2}{r^2} = 1 + e^2 + 2e\cos\theta$.

13. The difference is equal to $\dfrac{2}{SS'}$.

19. Make use of $SG = \dfrac{2SP\,.\,SS'}{SP + SS'}\cos\dfrac{PSS'}{2}$.

The locus is

$$\frac{l}{r} + \frac{(1+e)^2}{4e\cos\theta} - e\cos\theta = 0, \text{ a conic.}$$

20. The locus is

$$\frac{l}{r} + \frac{(1+e)^2}{2e\cos\theta} - (1+e)\cos\theta = 0.$$

XII. MISCELLANEOUS EXAMPLES ON THE ELLIPSE.

1. The equation of the tangent at any point, $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$, may be written

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\cos^2\theta-\frac{2x}{a}\cos\theta+1-\frac{y^2}{b^2}=0,$$

whence if $\cos\theta_1$, $\cos\theta_2$ are the roots

$$x_1x_2=a^2\cos\theta_1\cos\theta_2=a^2\left(1-\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{-1},$$

and we can obtain a similar relation for y_1y_2.

The locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$.

3. Pursuing the same method as in (1) put for x_1+x_2, x_2-x_1, &c. their values in $(x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2$. The locus is $xy=0$; the axes.

4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

6. If θ and ϕ be the eccentric angles at the extremities of one such pair of diameters, obtain and make use of the relation

$$\tan\theta\tan\phi=-\frac{a^2}{b^2}.$$

7. The conic (hyperbola)

$$x^2(k^2b^2+2a^2)-2b^2y^2=2(a^2-b^2)^2,$$

where k^2 is equal to the sum of the squares of the tangents of the inclinations.

8. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=r^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$.

12. Refer to CD and its conjugate as axes.

13. Let (α, β) be the given point, let θ and ϕ be the eccentric angles at two of the points where the circle meets the ellipse, and let (x, y) be the co-ordinates of the middle point between them. Then

$$(a \cos\theta - \alpha)^2 + (b \sin\theta - \beta)^2 = (a \cos\phi - \alpha)^2 + (b \sin\phi - \beta)^2,$$
$$2x = a(\cos\theta + \cos\phi),$$
$$2y = b(\sin\theta + \sin\phi).$$

Eliminating θ and ϕ between these three equations we obtain as the locus of (x, y)

$$a^2\alpha \,.\, y - b^2\beta \,.\, x = (a^2 - b^2)\, xy$$

which passes through the feet of the normals drawn from (α, β), being in fact the equation which, with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, determines them.

14. Take θ and ϕ as the points where the chord cuts the ellipse. The condition that it should subtend a right angle is found to be

$$\frac{k - b\sin\theta}{h - a\cos\theta} \cdot \frac{k - b\sin\phi}{h - a\cos\phi} + 1 = 0.$$

The locus is found to be

$$\frac{x^2}{a^2}(h^2 + k^2 - b^2) + \frac{y^2}{b^2}(h^2 + k^2 - a^2) - 2hx - 2ky + a^2 + b^2 = 0.$$

15. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + m^2.$

19. $x = \frac{a^4}{h(a^2 - b^2)}, \quad y = \frac{b^4}{k(b^2 - a^2)}.$

20. $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2.$

21. $(ma^2x - b^2y)(b^2x + ma^2y) = ma^2b^2(a^2 - b^2)$ if $y = mx$ be the diameter.

23. $y = -\frac{b}{a} \tan \alpha \,.\, x$ is the equation of the diameter if α is the eccentric angle of P. (See x. 43.)

27. The co-ordinates of the foci are

$$(0, 0), \left(-\frac{2e^2}{1-e^2} p \cos \alpha, \; -\frac{2e^2}{1-e^2} p \sin \alpha\right).$$

The directrices are

$$x \cos \alpha + y \sin \alpha = p, \quad x \cos \alpha + y \sin \alpha + p \,.\, \frac{1+e^2}{1-e^2} = 0.$$

XIII. THE PARABOLA REFERRED TO THE VERTEX

1. $am^2 + (2a - x) m - y = 0.$

3. (1) $y^2 = 2ax.$

(2) $y^2 = 2a (x - a).$

4. $y^2 = 2a (x - 4a).$

5. $\frac{y_1 y_2}{4a}, \; \frac{y_1 + y_2}{2}.$

6. $\frac{1}{8a} (y_1 - y_2) (y_2 - y_3) (y_3 - y_1).$

9. (1) $\sqrt{y^2 - 4ax} = \frac{x + 4a}{2}.$

(2) $y^2 - 4ax = (\sqrt{2} - 1)^2 (x + a)^2.$

12. $\pm \sqrt{2} y = x + 2 \sqrt{2}.$

13. The tangent at the vertex.

14. If $(am^2, 2am)$; $(an^2, 2an)$ are the co-ordinates of the points where the polar of (h, k) meets the curve we have to eliminate m and n between

$$\frac{(k-2am)(k-2an)}{(h-am^2)(h-an^2)}+1=0 \text{ ....(1)},$$

$$x = amn \text{ ....(2)},$$

$$y = a(m+n) \text{ ....(3)},$$

when we obtain the locus

$$x^2-\frac{h}{a}y^2+2(h+2a)x-2ky+h^2+k^2=0.$$

15. $4a^2(x+a)^2=c^2(4a^2+y^2)$ where c is the length of the perpendicular.

16. See (13).

24. $yy_1=2a(x+x_1)$ must be identical with

$$(x-a)\cos\theta+y\sin\theta=2a,$$

the tangent to the circle, which gives two equations; eliminating θ between them, we obtain the locus $y^2=(x+3a)(x-a)$.

25. $y^2+(x-a)(x+3a)=0$.

34. For Δ_3 see XIII. 8.

36. If m_1 and m_2 be the co-ordinates at P and Q,

$$SP=a(1+m_1^2), \quad SQ=a(1+m_2^2).$$

The loci are

(1) $y^2-2ax=a(k-2a)$.

(2) $(x-a)^2+y^2=k^2$.

37. To obtain the equation of the focus and directrix of a parabola

$$(ax-by)^2+2gx+2fy+c=0 \text{ .... (1)}.$$

The equation of the directrix must be of form $bx + ay + l = 0$ since it is perpendicular to all diameters; therefore if (m, n) be the focus, we must make (1) identical with

$$(x-m)^2 + (y-n)^2 = \frac{(bx+ay+l)^2}{a^2+b^2},$$

which will give three equations to find m, n and l.

(1) $(1, 2)$; $x + y - 3$.

(2) $(2, -3)$; $2x + y - 4$.

(3) $(3, 4)$; $2x + 3y + 4$.

42. Let $(an^2, 2an)$ denote the point where the normal meets the curve again. Substituting for x and y in the equation

$$am^3 + (2a - x)\,m - y = 0$$

of the normal, we obtain

$$am^3 + a\,(2 - n^2)\,m - 2an = 0;$$

and rejecting the factor $m - n$, we obtain the n of the second point.

The co-ordinates are

$$\frac{1}{4a}\left(\frac{8a^2+y_1^2}{y_1}\right)^2,\ -\left(\frac{8a^2+y_1^2}{y_1}\right),$$

and the length of the intercept

$$\frac{2}{y_1^2}\{4a^2+y_1^2\}^{\frac{3}{2}}.$$

45. See XIII. 1.

46. If h and k are the co-ordinates of the intersection of the tangents

$$x = \frac{k^2 + 2a^2 - ah}{a},\ y = -\frac{hk}{a}.$$

47. $y^2 = a\,(x - 2a)$.

48. Apply XIII. 1 and XIII. 42.

50. If the normal at P (x_1y_1) meet the tangent at the vertex in N and the tangent at P meet the axis in R, then the quadrilateral will be $APNR$, A being the vertex.

51. Apply XIII. 1.

52. In XIII. 1. The coefficient of m^2 is zero, therefore $m_1+m_2+m_3=0$, if m_1, m_2, m_3 be the roots.

53. Finding the ordinates of intersection of a circle with a parabola, we obtain a biquadratic without a term in y^3, therefore $y_1+y_2+y_3+y_4=0$; hence by (52), $y_4=0$; therefore the fourth point of intersection is the vertex.

58. The normal to $y^2-4ax=0$, namely

$$am^3+(2a-x)\,m-y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

must also be a normal to $x^2+2y^2=54a^2$; we obtain conditions $m=0$ which gives the common axis of the two curves, and $(m^2+2)^3=27$, which gives by substitution in (1) the remaining real lines satisfying the conditions, $\pm y+x=3a$.

XIV. THE PARABOLA REFERRED TO TWO TANGENTS AS AXES.

1. The axis is a line through the focus parallel to the line joining the origin to the middle point of the chord of contact $\left(\frac{a}{2}, \frac{b}{2}\right)$. Its equation is,

$$a\left(y-\frac{a^2b}{a^2+b^2}\right)=b\left(x-\frac{ab^2}{a^2+b^2}\right).$$

2. $by+ax=0$.

For the foot of the directrix

$$x=-\frac{ab^2(a^2-b^2)}{(a^2+b^2)^2},\quad y=\frac{a^2b\,(a^2-b^2)}{(a^2+b^2)^2}.$$

3. $x = \dfrac{ab^4}{(a^2+b^2)^2}$, $y = \dfrac{ba^4}{(a^2+b^2)^2}$.

4. The parameter is four times the length of the perpendicular let fall from the vertex on the directrix

$$= \frac{4a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}.$$

6. $x^2 + y^2 = by$, if AB is b.

11. Substitute the co-ordinates of the focus in the equation of the circle, which leads to the condition

$$\frac{h}{a} + \frac{k}{b} = 1.$$

12. $$(x^2 + y^2)\left(\frac{1}{x} + \frac{1}{y}\right) = k;$$

where k is the constant length.

13. $\dfrac{y}{x} = k^2$ where k is the constant ratio $a : b$.

14. $x^2 + y^2 = hx + ky$.

15. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \{hx^{\frac{1}{3}} + ky^{\frac{1}{3}}\}^{\frac{1}{2}}$.

16. Let OP, OP' be the tangents, S the focus; draw SM parallel to OP. Then the triangles OMS, OSP are similar.

17. Let $$\frac{x}{l} + \frac{y}{m} = 1 \text{ ....(1)}$$

be the equation of the directrix, where l and m have to be determined. Then l and m are the intercepts on the axes.

Let $\dfrac{x}{\sqrt{ax_1}} + \dfrac{y}{\sqrt{by_1}} = 1$ be a tangent intersecting the axis of y at the point where the directrix meets it. Then

$$m = \sqrt{by_1} \text{ ....(2).}$$

But the tangent must cut the axis of y at right angles since it meets it on the directrix,

$$\therefore \sqrt{by_1} = \sqrt{ax_1} \cos \omega,$$

from the condition of perpendicularity; whence

$$\frac{\sqrt{\dfrac{y_1}{b}}}{\dfrac{\cos \omega}{b}} = \frac{\sqrt{\dfrac{x_1}{a}}}{\dfrac{1}{a}} = \frac{ab}{b + a\cos\omega} \quad \text{..................(3)},$$

$$\therefore \sqrt{by_1} = \frac{ab\cos\omega}{b + a\cos\omega} = m.$$

Similarly we can find l; and substituting in (1) we obtain the equation required.

XVI. THE PARABOLA REFERRED TO THE FOCUS AS POLE AND THE AXIS AS INITIAL LINE.

2. $r = a \sec \dfrac{\alpha + \beta}{2} . \sec \dfrac{\alpha - \beta}{2}$.

3. $r = \dfrac{2a\cos(\theta - \alpha)}{\cos \dfrac{\alpha+\beta}{2} . \cos\dfrac{\alpha - \beta}{2}}$.

5. $\theta = \alpha$ and in either case therefore

$$r = \frac{2a}{\cos\theta + \cos\beta} \quad \text{..........................(1)}.$$

β can be eliminated between (1) and the condition.

The loci are (1) $r^2 \sin^2\theta + 2ar\cos\theta = k$,

(2) $r = c$,

where k and c are constants, that is, the loci are a parabola and a circle respectively.

9. $\left\{\dfrac{8a^2 \sin^2 \alpha}{(1 \pm \sin \alpha)^2 (1 \mp \sin \alpha)}\right\}^{\frac{1}{2}}$.

11. Two real intersections. The second curve is written

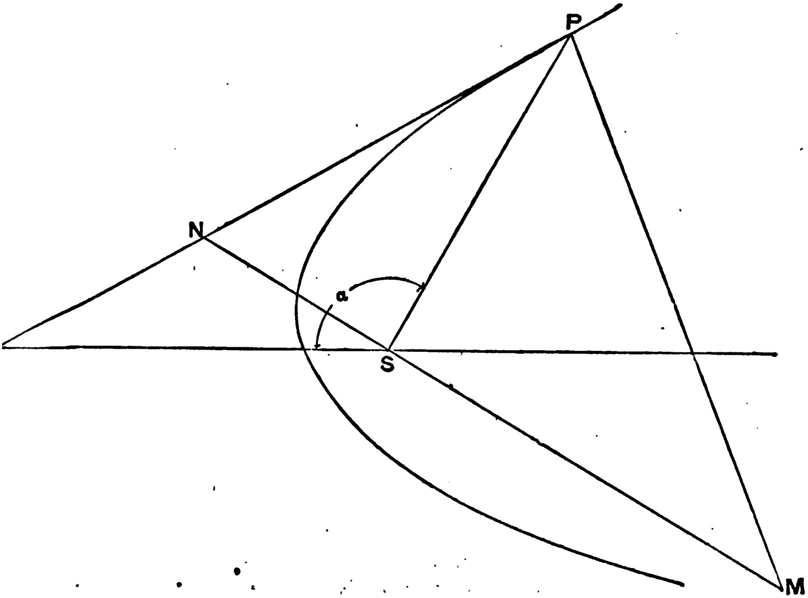
$$\frac{2b}{r} = 1 + \cos(\theta + \alpha).$$

12. Make the equations of the tangents to each curve identical.

15. $r \sin^2 \theta = a \cos \theta$.

16. Let PM be the normal; to find the locus of M. Let $SM = r$; then

$$\theta = \alpha + \frac{\pi}{2} \quad \text{..............................(1)},$$



also $$SP^2 = SN \,.\, r \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where N is the point where SM meets the tangent,

$$SP = \frac{2a}{1+\cos\alpha} = \frac{2a}{1+\sin\theta} \quad\ldots\ldots\ldots\ldots(3),$$

$$SN = 2a \operatorname{cosec}\alpha = -2a\sec\theta \quad\ldots\ldots\ldots\ldots(4);$$

substitute in (2).

17. $r = \dfrac{a\sec\theta}{1+\sin\theta}$.

18. $r = \dfrac{2a}{(1+\cos\theta)\left(1+2\sin\dfrac{\theta}{2}\right)}$.

20. $\dfrac{2a\sec\alpha}{r} = 1 + \sec\alpha \,.\, \cos\theta$.

21. $\dfrac{l}{r} = 1+\cos\theta$, where l is the length of the string.

XVII. THE HYPERBOLA REFERRED TO THE CENTRE AS ORIGIN.

4. $x = \dfrac{a\cos\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\alpha+\beta}{2}}$, $y = b\tan\dfrac{\alpha+\beta}{2}$.

5. $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2}$.

7. Use the equation of the tangent

$$x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}.$$

(1) $x^2 + y^2 = \sqrt{a^2x^2 - b^2y^2}$.

(2) $x(x \pm ae) + y^2 = a\sqrt{(x \pm ae)^2 - (e^2-1)y^2}$.

10. $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{a^2 + b^2}$.

13. $\pm 2y = x + 4a$.

Make the tangent to the parabola intersect the other curve in two coincident points.

17. The polar of any point $(a \sec\theta, b \tan\theta)$ with respect to the circle $(x - ae)^2 + y^2 = b^2$ is $a(x - ae)(\sec\theta - e) + yb\tan\theta = b^2$. Shew that this touches the circle described on the major axis.

20. $$(a^2 + b^2)^2 \tan^4\theta - 2by\,(a^2 + b^2)\tan^3\theta$$
$$+ \{b^2y^2 - a^2x^2 + (a^2 + b^2)^2\}\tan^2\theta - 2by\,(a^2 + b^2)\tan\theta$$
$$+ b^2y^2 = 0.$$

22. The perpendicular is $x = y \sin\theta$, which gives only imaginary points of intersection with $x^2 - y^2 = a^2$.

23. Take $OA = b$, $OC = a$; the equation of the hyperbola is $x^2 - y^2 + by - ax = 0$. It must be remembered that $x \pm y = 0$ represent two lines at right angles whatever be the inclination of the co-ordinate axes.

27. $(a^2 - k^2)x^2 + b^2y^2 - (a^2 + b^2)^2 = 0$ where k is a constant.

28. Let (x_1, y_1) be the co-ordinates of P, (x_2, y_2) of Q. The problem reduces to shewing that $x_1^2 + y_1^2 + x_1x_2 + y_1y_2 = 0$, which follows from the equations

$$x_1^2 - y_1^2 = x_2^2 - y_2^2, \quad \frac{x_2}{x_1} + \frac{y_2}{y_1} = 2.$$

The last equation expresses the fact that Q lies on the normal at P.

29. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where $2a$ and $2b$ are the axes of the ellipse.

30. The tangent at the point $(a, 0)$ is $x + y = a$.

XVIII. The Hyperbola referred to the Asymptotes as Axes.

2. (1) $\dfrac{2x_1x_2}{x_1+x_2}, \dfrac{2y_1y_2}{y_1+y_2}$.

(2) $\dfrac{x_1^2+x_1x_2+x_2^2+y_1y_2}{x_1+x_2}, \dfrac{y_1^2+y_1y_2+y_2^2+x_1x_2}{y_1+y_2}$.

3. $y^2+\dfrac{a}{2}x=0$.

6. $xy=\dfrac{k^4}{\alpha\beta}$.

11. $xy=\pm hmy+k(m+1)x$ where (h, k) is the fixed point.

12. $xy\sin\omega=\dfrac{mn}{(m+n)^2}kr$, where ω is the angle between the axes, k the constant perimeter, r the radius of the circle, $m:n$ the ratio in which the point divides AB.

XIX. The Hyperbola referred to Polar Co-ordinates, the Focus being the Pole.

2. The other asymptote.

4. (1) $(r-k)\left(\dfrac{\cos\theta}{a}\pm\dfrac{\sin\theta}{b}\right)=e$.

(2) $r=\dfrac{k^2}{e}\left(\dfrac{\cos\theta}{a}\pm\dfrac{\sin\theta}{b}\right)$; a circle through the focus.

7. The parameter of the parabola is $\dfrac{4b^2}{\sqrt{a^2+b^2}}$, the point of contact is given by $\alpha=2\tan^{-1}\left(\pm\dfrac{a}{b}\right)$.

11. $\dfrac{k}{r}=1+\mu\cos\phi$.

XX. THE GENERAL EQUATION OF THE SECOND DEGREE.

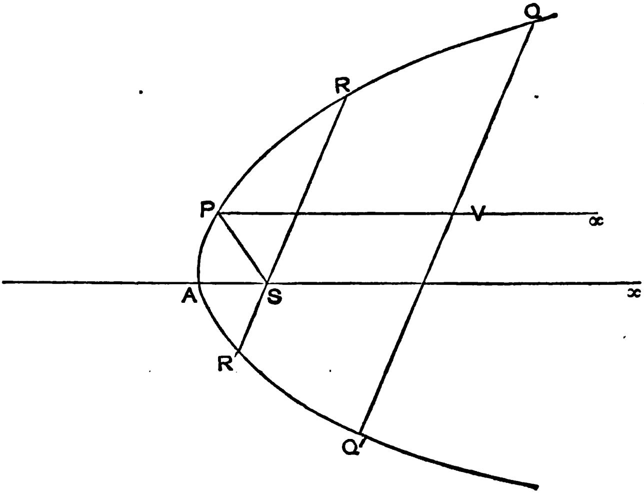
1. (1) $a=b$, $h=0$. (2) $h^2=ab$. (3) $a=-b$, since the asymptotes are at right angles.

2. The co-ordinates become infinite in value, which shews that all diameters of a parabola are parallel.

3. Apply the note preceding the example.

5. To shew that $QV^2=4SP.PV$.

Through the focus S draw RSR' parallel to QQ' and let PV be the diameter to QQ'. Join SP. Let ∞ denote the points at infinity on AS and PV.



Then
$$\frac{QV.VQ'}{PV.V\infty}=\frac{RS.SR'}{AS.S\infty},$$

whence
$$QV^2=PV.\frac{RS.SR'}{AS}.$$

It is left as an exercise for the student to shew that

$$RS \cdot SR' = 4SA \cdot SP,$$

which he can do either geometrically or by applying the polar equation of the parabola referred to the focus.

6. If θ be the half angle between the asymptotes

$$\cos\theta = -\frac{1}{e},$$

and 2θ is the angle between the lines represented by

$$ax^2 + 2hxy + by^2 = 0.$$

8. $(ax + hy + g)^2 + (hx + by + f)^2 = k(gx + fy + c)^2$, a conic.

10. $c + d = 0$.

11. The radical axis of the two circles.

12. If r be the radius of a circle, l the length of the perpendicular drawn from the centre on a straight line, the length of the intercept on the line is $2\sqrt{r^2 - l^2}$.

The result is

$$2\{g^2\sin^2\alpha + f^2\cos^2\alpha - fg\sin 2\alpha - c - p(2g\cos\alpha + 2f\sin\alpha + p)\}^{\frac{1}{2}}.$$

13. $r\cos\alpha = p$.

14. Take the general equation of the circle to be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ .... } (1).$$

Let (x_1, y_1) (x_2, y_2) be two points on the circle that subtend a right angle at the origin. Then

$$x_1x_2 + y_1y_2 = 0 \text{ .... } (2).$$

Substitute from $x\cos\alpha + y\sin\alpha - p = 0$ in (1) first for x, then for y; this will give the two quadratics, finding the ordinates and abscissae respectively; y_1y_2, x_1x_2 will be the absolute terms in these equations, whence

$$y_1y_2 + x_1x_2 = 2p^2 + 2p(g\cos\alpha + f\sin\alpha) + c = 0.$$

15. Let (x_1, y_1) (x_2, y_2) be the co-ordinates of the extremities of the chord; let (x, y) be the middle point.

Then $$x=\frac{x_1+x_2}{2},\quad y=\frac{y_1+y_2}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

$$\left.\begin{aligned}x_1^2+y_1^2+2gx_1+2fy_1+c&=0\\ x_2^2+y_2^2+2gx_2+2fy_2+c&=0\\ 2x_1x_2+2y_1y_2&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(2);$$

adding the equations in (2) and substituting from (1) the locus is found to be the circle

$$x^2+y^2+gx+fy+\frac{c}{2}=0.$$

Compare Salmon, pp. 90, 91.

16. $\frac{a}{l}+\frac{b}{m}=a^2+b^2.$

17. $a=b=\frac{1}{2}.$

18. The condition becomes identical with the one that makes them confocal.

19. $b^2xy=(bx+ay-ab)(bx+ay-bc).$

20. Let $OA=a$, $OB=b$. The locus is $ay=bx$.

21. $y=-mx.$

23. $\frac{1}{ab'}+\frac{1}{a'b}+\left(\frac{1}{aa'}+\frac{1}{bb'}\right)\sec\omega.$

24. $x^2+y^2-\frac{a+a'}{2}x-\frac{b+b'}{2}y=0;$

a circle through the origin, and through points on the axes, whose distances from the origin are arithmetic means between the intercepts of the lines.

25. Taking the sides a, b as axes the locus is

$$x^2+y^2=\frac{ax+by}{2}.$$

26. $\frac{\pi\,.\,c}{\sqrt{(ab-h^2)}}.$

27. $\frac{\pi\,.\,(abc+2fgh-af^2-bg^2-ch^2)}{(ab-h^2)^{\frac{3}{2}}}.$

29. Let (m, n) be the co-ordinates of the focus. The directrix must be perpendicular to all diameters; it is therefore of the form $\beta x - \alpha y + k = 0$. The parabola may therefore be written in the form

$$(x-m)^2 + (y-n)^2 = \frac{(\beta x - \alpha y + k)^2}{\alpha^2 + \beta^2}$$

which reduces to

$$(\alpha x + \beta y)^2 - 2x\left[(\alpha^2+\beta^2)\,m + \beta k\right] - 2y\left[(\alpha^2+\beta^2)\,n - \alpha k\right] + (m^2+n^2)(\alpha^2+\beta^2) - k^2 = 0.$$

Making this form of the equation identical with

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0,$$

we obtain k and the values of m and n.

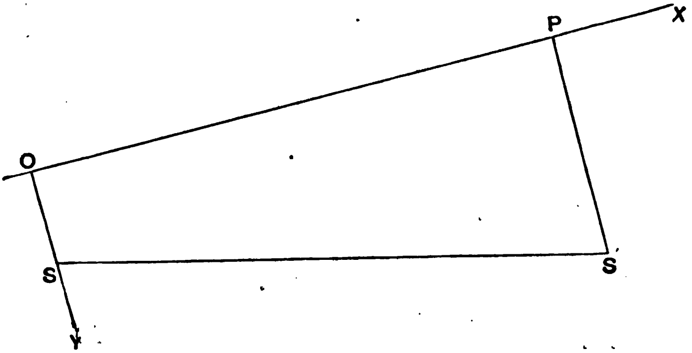
The student is recommended to deduce the corresponding values for

$$\frac{x^2}{a^2} - \frac{2xy}{ab} + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0,$$

which have already been investigated in another way. (See xiv.)

31. Let S be the given focus, OP the given tangent: it is required to find the locus of S'. Let $OS = p$, then taking OS and OP as axes of y and x respectively, we have to eliminate a between $py = b^2 = a^2(1-e^2)$

and $x^2 + (y-p)^2 = 4a^2e^2$.



XXI. ABRIDGED NOTATION.

1. The tangent at A has been shewn to be $m\beta+n\gamma=0$. Take any point P in the tangent at A and draw perpendiculars PM, PN on AB and AC respectively. Then

$$\frac{PM}{PN}=\frac{n}{m}=\frac{\sin C}{\sin B}$$

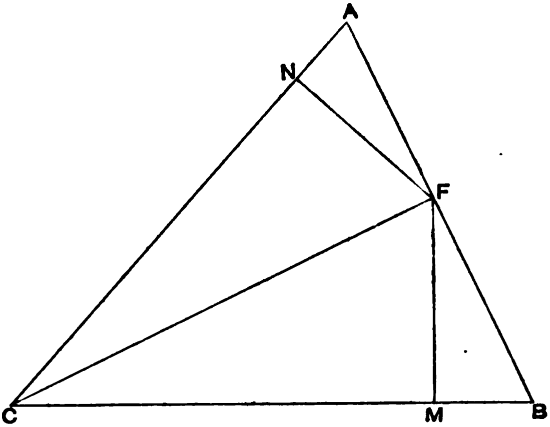
and from the other tangents we deduce similar relations;

$$\therefore\ l : m : n :: \sin A : \sin B : \sin C,$$

and the circumscribing circle is

$$\sin A \,.\, \beta\gamma+\sin B \,.\, \gamma\alpha+\sin C \,.\, \alpha\beta=0.$$

2. By (IV.) we see that $l\alpha-m\beta=0$ is the line joining C to F the opposite point of contact,



$$\therefore\ \frac{l}{m}=\frac{FN}{FM}=\frac{FA\,.\,\sin A}{FB\,.\,\sin B}$$

$$=\frac{\cot\frac{A}{2}\,.\,\sin A}{\cot\frac{B}{2}\,.\,\sin B}=\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}.$$

The equation of the inscribed circle therefore is

$$\cos\frac{A}{2}\sqrt{\alpha}+\cos\frac{B}{2}\sqrt{\beta}+\cos\frac{C}{2}\sqrt{\gamma}=0.$$

3. By the method used in (2) we obtain

$$l:m:n::a:b:c;$$

$\therefore$ the ellipse required is

$$\sqrt{a\alpha}+\sqrt{b\beta}+\sqrt{c\gamma}=0.$$

4. (1) $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}=0.$

(2) $\sqrt{\dfrac{\alpha}{\cos A}}+\sqrt{\dfrac{\beta}{\cos B}}+\sqrt{\dfrac{\gamma}{\cos C}}=0.$

5. Assume the circle to be

$$l\alpha^2+m\beta^2+n\gamma^2=0,$$

then writing for α, $x\cos a+y\sin a-p$; and similar expressions for β and γ; we obtain for the coefficient

of x^2 $\quad l\cos^2 a+m\cos^2\beta+n\cos^2\gamma$..................(1),

of y^2 $\quad l\sin^2 a+m\sin^2\beta+n\sin^2\gamma$..................(2),

of xy $\quad l\sin 2a+m\sin 2\beta+n\sin 2\gamma$ ....(3),

equating (1) and (2) and putting (3) equal to zero we have

$$l\cos 2a+m\cos 2\beta+n\cos 2\gamma=0,$$
$$l\sin 2a+m\sin 2\beta+n\sin 2\gamma=0;$$

whence $\dfrac{l}{\sin 2(\beta-\gamma)}=\dfrac{m}{\sin 2(\gamma-a)}=\dfrac{n}{\sin 2(a-\beta)}$;

now $\beta-\gamma=\pi-A$, &c.

$$\therefore\ l:m:n::\sin 2A:\sin 2B:\sin 2C,$$

and the circle is $\sin 2A\alpha^2+\sin 2B\beta^2+\sin 2C\gamma^2=0.$

6. (1) Let $l\alpha+m\beta+n\gamma=0$ be any line through $(\alpha_1, \beta_1, \gamma_1)$, then

$$\frac{l}{\beta\gamma_1-\gamma\beta_1}=\frac{m}{\gamma\alpha_1-\alpha\gamma_1}=\frac{n}{\alpha\beta_1-\beta\alpha_1};$$

(α, β, γ) being any point in the line.

So if $l'\alpha + m'\beta + n'\gamma = 0$ pass through $(\alpha_2, \beta_2, \gamma_2)$ we have

$$\frac{l'}{\beta\gamma_2 - \gamma\beta_2} = \frac{m'}{\gamma\alpha_2 - \alpha\gamma_2} = \frac{n'}{\alpha\beta_2 - \beta\alpha_2}.$$

If these lines intersect at right angles at (α, β, γ) we must substitute these values for $l : m : n$, and $l' : m' : n'$ in the condition of perpendicularity

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0,$$

when we obtain the equation of the circle required if we make (α, β, γ) current co-ordinates.

(2) $\alpha\beta = \gamma(\gamma\cos C - \alpha\cos A - \beta\cos B)$.

8. $a\alpha + b\beta + c\gamma$ being a constant, the terms involving second powers of the variables are left unaltered, therefore the equation represents a circle; and it represents *any* circle, since we may give to l, m, n any values, that is to say we may make the circle satisfy three conditions which completely determine it.

Writing two circles in the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma),$$
$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l'\alpha + m'\beta + n'\gamma),$$

their radical axis is

$$(l - l')\alpha + (m - m')\beta + (n - n')\gamma = 0.$$

9. $\alpha\cos A = \gamma\cos C$.

15. Apply (14).

16. $k = -2a$.

18. $a = b = c$.

22. Make $\alpha^2 = k\beta\gamma$ touch the line at infinity.

25. $\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0$.

26. The general equation

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta = 0$$

may be written

$$k(a\beta\gamma + b\gamma\alpha + c\alpha\beta) + (a\alpha + b\beta + c\gamma)\left(\frac{A}{a}\cdot\alpha + \frac{B}{b}\cdot\beta + \frac{C}{c}\cdot\gamma\right) = 0.$$

See XXI. 8.

Equating coefficients of $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ we obtain

$$\begin{aligned} Ab^2 + Ba^2 - 2abH &= Bc^2 + Cb^2 - 2bcF \\ &= Ca^2 + Ac^2 - 2caG, \end{aligned}$$

as the conditions required.

27. Make the tangent at $(\alpha_1, \beta_1, \gamma_1)$,

$$(A\alpha_1 + H\beta_1 + G\gamma_1)\,\alpha + (H\alpha_1 + B\beta_1 + F\gamma_1)\,\beta + (G\alpha_1 + F\beta_1 + C\gamma_1)\,\gamma = 0,$$

identical with the line at infinity.

The result is expressed by the determinant

$$\begin{vmatrix} A & H & G & a \\ H & B & F & b \\ G & F & C & c \\ a & b & c & 0 \end{vmatrix} = 0.$$

28. Let α_0, β_0, γ_0 be the co-ordinates of the centre of the conic.

Let $\frac{\alpha - \alpha_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\nu} = r$ be the equation of an asymptote.

Substituting

$$\alpha = \alpha_0 + \lambda r,\ \beta = \beta_0 + \mu r,\ \gamma = \gamma_0 + \nu r \text{ in } f(\alpha, \beta, \gamma) = 0,$$

we obtain an equation in r of the form

$$Pr^2 + Qr + S = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1),$$

where P is found to be $f(\lambda, \mu, \nu)$, i. e. the left-hand member of the general equation of the second degree with λ, μ, ν substituted for the current co-ordinates; Q and S being functions of α_0, β_0, γ_0; λ, μ, ν.

Writing (1) in the form

$$P + \frac{Q}{r} + \frac{S}{r^2} = 0,$$

and making r infinite since it represents an asymptote, we obtain $P=0$, or $f(\lambda, \mu, \nu)=0$; an equation of the second degree which determines the direction-cosines of the asymptotes.

29. Writing $\cos \alpha$, $\cos \beta$, $\cos \gamma$ for λ, μ, ν, we have by (28)

$$A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma + 2F \cos \beta \cos \gamma + 2G \cos \gamma \cos \alpha$$
$$+ 2F \cos \alpha \cos \beta = 0 \text{ .... (1).}$$

But since the asymptotes are at right angles, (1) must be true if for α, β, γ, we write

$$\frac{\pi}{2} + \alpha,\ \frac{\pi}{2} + \beta,\ \frac{\pi}{2} + \gamma.$$

Adding the result of substitution to (1) and remembering that $\beta - \gamma = \pi - A$, &c., we get the condition required.

30. The condition that the curve should be an equilateral hyperbola will be found to be the same as that which makes it pass through the point $\sec A : \sec B : \sec C$.

33. $\alpha^2 - \beta^2 = k(\gamma^2 - \alpha^2)$.

34. The conic must be of the form

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0,$$

also of the form

$$(l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma) = (a\alpha + b\beta + c\gamma)^2,$$

and must be an equilateral hyperbola which gives another relation.

These equations will give the condition required, and will shew that the other asymptote is

$$\frac{a^2}{l}\alpha + \frac{b^2}{m}\beta + \frac{c^2}{n}\gamma = 0.$$

36. The other asymptote is

$$l\left(\frac{n}{c} + \frac{m}{b} - \frac{l}{a}\right)\alpha + m\left(\frac{l}{a} + \frac{n}{c} - \frac{m}{b}\right)\beta + n\left(\frac{m}{b} + \frac{l}{a} - \frac{n}{c}\right)\gamma = 0.$$

The condition that the hyperbola should be rectangular is

$$\frac{l}{bn - cm} + \frac{m}{cl - an} + \frac{n}{am - bl} = 0.$$

XXII. ENVELOPES.

1. $y^2 = \frac{k^2}{a}\left(x + \frac{k^2}{4a}\right).$

2. $xy = \frac{k^2}{4}.$

3. $x^2 - 2xy + y^2 - 2kx - 2ky + k^2 = 0.$

4. The parabola

$$\alpha^2y^2 = (b^2 - a^2)(2\alpha x + b^2 - a^2 - \alpha^2).$$

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}.$

6. Let the straight line be

$$x \cos \alpha + y \sin \alpha = p;$$

then p is found from the equation

$$2p^2 + 2p(g \cos \alpha + f \sin \alpha) + c = 0.$$

See xx. 14.

Substitute from this equation for p; when we obtain as the envelope of

$$x \cos \alpha + y \sin \alpha = \frac{-(g \cos \alpha + f \sin \alpha) \pm \sqrt{(g \cos \alpha + f \sin \alpha)^2 - 2c}}{2},$$

the equation

$$x^2 + y^2 = \frac{g^2 + f^2}{g^2 + f^2 - 2c}\left(\frac{gx + fy + c}{\sqrt{g^2 + f^2}}\right)^2.$$

7. $a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2ab\alpha\beta - 2bc\beta\gamma - 2ca\gamma\alpha = 0.$

8. The locus is the same as (7). Use $pa\alpha + qb\beta + rc\gamma = 0$ as the equation of the straight line.

9. $a^2\alpha^2 = b^2\beta^2 + c^2\gamma^2$, or one of the two similar equations.

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